TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 11, November 1996

COVERING THE INTEGERS BY ARITHMETIC SEQUENCES. II

ZHI-WEI SUN

ABSTRACT. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ ($n_1 \leqslant \cdots \leqslant n_k$) be a system of arithmetic sequences where $a_1, \cdots, a_k \in \mathbb{Z}$ and $n_1, \cdots, n_k \in \mathbb{Z}^+$. For $m \in \mathbb{Z}^+$ system A will be called an (exact) m-cover of $\mathbb{Z}$ if every integer is covered by A at least (exactly) m times. In this paper we reveal further connections between the common differences in an (exact) m-cover of $\mathbb{Z}$ and Egyptian fractions. Here are some typical results for those m-covers A of $\mathbb{Z}$: (a) For any $m_1, \cdots, m_k \in \mathbb{Z}^+$ there are at least m positive integers in the form $\Sigma_{s\in I} m_s/n_s$ where $I \subseteq \{1, \cdots, k\}$. (b) When $n_{k-l} < n_{k-l+1} = \cdots = n_k$ ($0 < l < k$), either $l \geqslant n_k/n_{k-l}$ or $\Sigma_{s=1}^{k-l} 1/n_s \geqslant m$, and for each positive integer $\lambda < n_k/n_{k-l}$ the binomial coefficient $\binom{l}{\lambda}$ can be written as the sum of some denominators > 1 of the rationals $\Sigma_{s\in I} 1/n_s - \lambda/n_k$, $I \subseteq \{1, \cdots, k\}$ if A forms an exact m-cover of $\mathbb{Z}$. (c) If $\{a_s + n_s\mathbb{Z}\}_{s=1, s\neq t}^k$ is not an m-cover of $\mathbb{Z}$, then $\Sigma_{s\in I} 1/n_s$, $I \subseteq \{1, \cdots, k\} \setminus \{t\}$ have at least n_t distinct fractional parts and for each $r = 0, 1, \cdots, n_t - 1$ there exist $I_1, I_2 \subseteq \{1, \cdots, k\} \setminus \{t\}$ such that $r/n_t \equiv \Sigma_{s\in I_1} 1/n_s - \Sigma_{s\in I_2} 1/n_s \pmod 1$. If A forms an exact m-cover of $\mathbb{Z}$ with $m = 1$ or $n_1 < \cdots < n_{k-l} < n_{k-l+1} = \cdots = n_k$ ($l > 0$) then for every $t = 1, \cdots, k$ and $r = 0, 1, \cdots, n_t - 1$ there is an $I \subseteq \{1, \cdots, k\}$ such that $\Sigma_{s\in I} 1/n_s \equiv r/n_t \pmod 1$.

1. Background and Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers and $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ the set of positve integers. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we call the infinite set

$$a + n\mathbb{Z} = \{a + jn\}_{j=-\infty}^{+\infty} = \{\cdots, a - 2n, a - n, a, a + n, a + 2n, \cdots\}$$

an *arithmetic sequence* with common difference n (abbreviated to $AS(n)$) and its finite subsequence

$$\{a + jn\}_{j=m}^{l+m-1} = \{a + mn, a + (m+1)n, \cdots, a + (l+m-1)n\}$$

an *arithmetic progression* of length l with common difference n (in short, $AP_n(l)$).

Let m be a positive integer. A finte system

$$A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k \qquad (a_1, \cdots, a_k \in \mathbb{Z} \text{ and } n_1, \cdots, n_k \in \mathbb{Z}^+) \tag{1}$$

of arithmetic sequences is said to be an (*exact*) *m-cover of* $S \subseteq \mathbb{Z}$ if

$$|\{1 \leqslant s \leqslant k : \ x \in a_s + n_s\mathbb{Z}\}| \geqslant m \quad (\text{resp. } |\{1 \leqslant s \leqslant k : \ x \in a_s + n_s\mathbb{Z}\}| = m)$$

for all $x \in S$. Instead of the term '1-cover' we simply use the word 'cover'.

Received by the editors June 7, 1994 and, in revised form, November 10, 1995.
1991 *Mathematics Subject Classification.* Primary 11B25; Secondary 11A07, 11B75, 11D68.
This research is supported by the National Natural Science Foundation of P. R. China.

For any $n \in \mathbb{Z}^+$, $\{r+n\mathbb{Z}\}_{r=0}^{n-1}$ is an exact cover of $\mathbb{Z}$. In this trivial cover of $\mathbb{Z}$ all the common differences are equal. Clearly m (exact) covers of $\mathbb{Z}$ form together an (exact) m-cover of $\mathbb{Z}$. It is known that for each $m = 2, 3, 4, \cdots$ there exists an exact m-cover of $\mathbb{Z}$ no subcover of which is an exact n-cover of $\mathbb{Z}$ with $0 < n < m$ (cf. [26]). Observe that if A is an m-cover of S then so is A together with $a + n\mathbb{Z}$ where $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. If (1) is an m-cover of S but $\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^{k}$ is not, then we say that (1) forms an m-cover of S with $a_t + n_t\mathbb{Z}$ *essential*. A *minimal* m-cover is an m-cover in which all the arithmetic sequences are essential.

At the beginning of the thirties P. Erdös ([6]) introduced the concept of cover of $\mathbb{Z}$ (by arithmetic sequences) and gave a nontrivial example

$$\{2\mathbb{Z},\ 3\mathbb{Z},\ 1+4\mathbb{Z},\ 3+8\mathbb{Z},\ 7+12\mathbb{Z},\ 23+24\mathbb{Z}\}$$

which was used to show that there are infinitely many odd positive integers not of the form $2^k + p$ with $k \geqslant 1$ and p an odd prime. Since then covers of $\mathbb{Z}$ have been investigated by many authors. A central problem in this area is to characterize the common differences in an arbitrary (exact) m-cover of $\mathbb{Z}$.

By a simple cardinality argument, $\Sigma_{s=1}^k 1/n_s \geqslant m$ if (1) is an m-cover of $\mathbb{Z}$, and (1) is an exact m-cover of $\mathbb{Z}$ if and only if it's an m-cover of $\mathbb{Z}$ with $\Sigma_{s=1}^k 1/n_s = m$. Observe that the arithmetic sequences in an exact cover (of $\mathbb{Z}$) must be pairwise disjoint, so a necessary condition for (1) to be an exact cover of $\mathbb{Z}$ is that $(n_s, n_t) > 1$ if $1 \leqslant s < t \leqslant k$. (In this paper $(m_1, \cdots, m_k)$ or $(m_i)_{i=1}^k$, and $[m_1, \cdots, m_k]$ or $[m_i]_{i=1}^k$ stand for the greatest common divisor and the least common multiple of not all zero integers $m_1, \cdots, m_k$ respectively.) Besides these trivial things what more can we say about the common differences in an m-cover of $\mathbb{Z}$? Here are three main conjectures the second of which implies the third one (cf. [8], [14]).

I. Erdös' Conjecture. *For any $c > 0$ there always exists a cover of $\mathbb{Z}$ with all the common differences distinct and greater than c.*

II. The Erdös-Selfridge Conjecture. *If (1) forms a cover of $\mathbb{Z}$ with all the n_s distinct and greater than one, then n_s is even for some $s = 1, \cdots, k$.*

III. Schinzel's Conjecture. *Providing (1) is a cover of $\mathbb{Z}$, $n_s \mid n_t$ for some $s, t = 1, \cdots, k$ with $s \neq t$.*

We mention that under the Erdös−Selfridge conjecture for each polynomial $P(x) \in \mathbb{Z}[x]$ with $P(0) \neq 0$, $P(1) \neq -1$ and $P(x) \not\equiv 1$ there exist infinitely many $n \in \mathbb{Z}^+$ such that $x^n + P(x)$ is irreducible over the rational field $\mathbb{Q}$. This nice application was first noted by A. Schinzel [14].

In [22] Z. W. Sun and Z. H. Sun showed that if $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ is a cover of $\mathbb{Z}$ but $\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ d\nmid n_s}}^{k}$ is not then

$$|\{1 \leqslant s \leqslant k:\ d \mid n_s\}| \geqslant d/(d, [n_s]_{\substack{0\leqslant s\leqslant k\\ d\nmid n_s}}) \tag{2}$$

where d is any integer greater than $n_0 = 1$, thus when (1) forms a minimal cover of $\mathbb{Z}$ for any prime power p^α dividing some of $n_1, \cdots, n_k$ we have

$$|\{1 \leqslant s \leqslant k:\ p^\alpha \mid n_s\}| \geqslant p^\delta$$

where δ is the smallest positive integer such that $p^{\alpha-\delta}$ divides one of those $n_0 = 1,\ n_1, \cdots, n_k$ not divisible by p^α. In 1966 Š. Znám [27] confirmed the following conjecture of J. Mycielski: If $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ forms an exact cover of $\mathbb{Z}$, then

$k \geqslant 1 + f(n_s)$ for each $s = 1, \cdots, k$ where the Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N}$ is defined by

$$f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) = \sum_{i=1}^{r} \alpha_i(p_i - 1) \quad \text{where } p_1 < p_2 < \cdots < p_r \text{ are primes.}$$

In 1975 Znám [29] conjectured further that $k \geqslant 1 + f([n_1, \cdots, n_k])$ providing (1) is a minimal cover of $\mathbb{Z}$, this was confirmed by R. J. Simpson [15] in 1985. The strongest result in this direction appeared in Theorem 11 of Z. W. Sun [17]: Let (1) be a minimal m-cover of $\mathbb{Z}$ and d a divisor of $N = [n_1, \cdots, n_k]$ with $0 < d < N$, then

$$|\{1 \leqslant s \leqslant k :\ n_s \nmid d\}| > f(N/d) \tag{3}$$

and moreover there exist $l = f(N/d)$ distinct positive integers $m_1, \cdots, m_l$ less than N/d and l distinct indices $i_1 < \cdots < i_l$ between 1 and k such that $n_{i_s} \nmid dm_s$ for every $s = 1, \cdots, l$. In [17] the following conjecture was proposed.

IV. Sun's Conjecture. *Let (1) be a minimal cover of $\mathbb{Z}$ with $n_0 = 1 < n_1 < \cdots < n_k$. Assume that $N = [n_1, \cdots, n_k]$ has the standard form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1 < \cdots < p_r$ are primes. Set*

$$\{\alpha \in \mathbb{N} :\ p_s^{\alpha} \| n_i \text{ for some } i = 0, 1, \cdots, k\} = \{\alpha_{s0}, \alpha_{s1}, \cdots, \alpha_{si_s}\}$$

with $1 \leqslant s \leqslant r$ and $\alpha_{s0} = 0 < \alpha_{s1} < \cdots < \alpha_{si_s} = \alpha_s$. (By $p^{\alpha} \| n$ we mean $p^{\alpha} \mid n$ and $p^{\alpha+1} \nmid n$.) Then

$$\begin{aligned} &|\{1 \leqslant i \leqslant k :\ n_i = p_1^{\beta_1} \cdots p_{s-1}^{\beta_{s-1}} p_s^{\alpha_{st}} \text{ for some } \beta_1 \leqslant \alpha_1, \cdots, \beta_{s-1} \leqslant \alpha_{s-1}\}| \\ &\qquad \geqslant (\alpha_{st} - \alpha_{s(t-1)})(p_s - 1) \end{aligned} \tag{4}$$

holds for all $s = 1, \cdots, r$ and $t = 1, \cdots, i_s$.

As

$$|\{1 \leqslant s \leqslant k :\ p_r^{\alpha_r} \mid n_s\}| \geqslant p_r^{\alpha_{ri_r} - \alpha_{r(i_r-1)}} \geqslant 1 + (\alpha_{ri_r} - \alpha_{r(i_r-1)})(p_r - 1)$$

and

$$f(N) = \sum_{s=1}^{r} \sum_{t=1}^{i_s} (\alpha_{st} - \alpha_{s(t-1)})(p_s - 1),$$

the assertion in Sun's conjecture implies that $k \geqslant 1 + f([n_1, \cdots, n_k])$. Observe that $p_1 = 2$ if $|\{1 \leqslant i \leqslant k :\ n_i = p_1^{\alpha_{11}}\}| \geqslant p_1 - 1$. So Conjecture II follows from Conjecture IV.

It should be mentioned that almost nothing is known about the four conjectures, among which the first and the second have been open for about fifty years.

Fortunately we have an already-proved conjecture of P. Erdös which asserts that $\Sigma_{s=1}^{k} 1/n_s > 1$ if (1) is a cover of $\mathbb{Z}$ with all the n_s distinct and greater than one (i.e. (1) cannot be an exact cover of $\mathbb{Z}$ if $1 < n_1 < \cdots < n_k$). In 1986 Simpson [16] confirmed a conjecture of N. Burshtein [2] by showing that if (1) is an exact cover of $\mathbb{Z}$ and $p_1 < p_2 < \cdots < p_r$ are the prime factors of $[n_1, \cdots, n_k] > 1$ then

$$\max_{1 \leqslant t \leqslant k} |\{1 \leqslant s \leqslant k :\ n_s = n_t\}| \geqslant p_r \prod_{i=1}^{r-1} \frac{p_i - 1}{p_i} > p_1 - 1 = p([n_1, \cdots, n_k]) - 1.$$

(Throughout this paper we use $p(n)$ to denote the least prime factor of an integer $n > 1$.) With the help of Merten's theorem it follows that for each $M \in \mathbb{Z}^+$ there exists a number $B(M)$ such that, in any exact cover of $\mathbb{Z}$ whose common differences are repeated at most M times, the least common difference is not more than $B(M)$. This can be viewed as a negative answer to an analogy of Conjecture I for exact covers of $\mathbb{Z}$. In 1991 Sun [18] gave a result stronger than the one of Simpson, in fact he proved that if $a_1+n_1\mathbb{Z},\cdots,a_k+n_k\mathbb{Z}$ are pairwise disjoint and the characteristic function of the set $\bigcup_{s=1}^k a_s+n_s\mathbb{Z}$ is periodic mod n_0 (which happen for $n_0=1$ if $\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms an exact cover of $\mathbb{Z}$) then for any $I\subseteq\{1,\cdots,k\}$ we have

$$\sum_{s\in I}\frac{1}{n_s}\geqslant d\left(\bigcup_{s\in I}(n_s,[n_t]_{\substack{0\leqslant t\leqslant k\\ t\notin I}})\mathbb{Z}\right) \tag{5}$$

where $d(S)$ stands for the asymptotic density

$$\lim_{N\to+\infty}\frac{1}{N}|\{0\leqslant x<N:\ x\in S\}|$$

of $S\subseteq\mathbb{Z}$ if the limit exists.

The last conjecture of Erdös was first confirmed independently by H. Davenport, L. Mirsky, D. Newman and R. Radó, who showed that $n_{k-1}=n_k$ providing (1) is an exact cover of $\mathbb{Z}$ with $1<n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$. Znám ([28]) conjectured in 1968 and M. Newman ([11]) proved in 1971 that $n_{k-1}=n_k$ above can be strengthened by $n_{k-p+1}=\cdots=n_k$ with $p=p(n_k)$. Among the various improvements to the Newman-Znám result, Theorem 1 of Sun [19] indicates that if the covering function

$$\sigma(x)=|\{1\leqslant s\leqslant k:\ x\in a_s+n_s\mathbb{Z}\}|$$

is periodic mod n_0 with $d\nmid n_0$ and n_s is divisible by d for some $s=1,\cdots,k$ then

$$|\{1\leqslant s\leqslant k:\ d\mid n_s\}|\geqslant\min_{\substack{0\leqslant s\leqslant k\\ d\nmid n_s}}\frac{d}{(d,n_s)}\geqslant p(d), \tag{6}$$

in particular if $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$ with

$$n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k \tag{7}$$

where $0<l<k$ then

$$l\geqslant\min_{1\leqslant s\leqslant k-l}\frac{n_k}{(n_s,n_k)}\geqslant\max\left\{p(n_k),\frac{n_k}{n_{k-l}}\right\}.$$

Due to the efforts of S. K. Stein, Znám, Š. Porubský, M. A. Berger, A. Felzenbaum and A. S. Fraenkel, the common differences in an arbitrary exact m-cover (1) of $\mathbb{Z}$ with

$$n_1<\cdots<n_{k-l}<n_{k-l+1}=\cdots=n_k \tag{8}$$

have been determined completely in the cases $m=1$ and $2\leqslant l\leqslant 9$, $m>1$ and $2\leqslant l\leqslant 5$ (cf. [1], [12]).

In contrast with the confirmed conjecture of Erdös, M. Z. Zhang ([25]) discovered in 1989 that there exists an $I\subseteq\{1,\cdots,k\}$ with $\Sigma_{s=1}^k 1/n_s\in\mathbb{Z}^+$ if (1) is a cover of $\mathbb{Z}$. In 1992 Sun [20] showed that when (1) forms an exact m-cover of $\mathbb{Z}$ for each $n=0,1,\cdots,m$ we have

$$\sum_{s\in I}\frac{1}{n_s}=n\qquad\text{for at least }\binom{m}{n}\text{ subsets }I\text{ of }\{1,\cdots,k\} \tag{9}$$

where the bounds $\binom{m}{n}$ $(0 \leqslant n \leqslant m)$ are best possible and as usual $\binom{x}{n}$ stands for $\prod_{j=0}^{n-1} \frac{x-j}{n-j}$. Both Sun and Zhang used the Riemann ζ-function in their proofs.

We have reviewed the main problems and results concerned with the common differences in an (exact) m-cover of $\mathbb{Z}$. With the background in mind the reader may realize that it's very difficult and quite fascinating to provide something new and general in this aspect.

In [21] we began our systematic investigation of m-covers of $\mathbb{Z}$ by a new approach rooted in [20]. We first employed some knowledge of analysis, algebra and combinatorics to characterize those m-covers $\mathcal{A} = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ of $\mathbb{Z}$ by generalized arithmetic sequences where $\alpha_1, \cdots, \alpha_k$ are reals and $\beta_1, \cdots, \beta_k$ are positive reals, and then obtained several necessary conditions on the β's from the characterizations. Of course such results apply to those m-covers (1) of $\mathbb{Z}$ by usual arithmetic sequences. A remarkable one following from Theorem 4 of [21] is that if $\emptyset \subset J \subset \{1, \cdots, k\}$, then

$$\sum_{s\in I} \frac{1}{n_s} = \sum_{s\in J} \frac{1}{n_s} \qquad \text{for some } I \subseteq \{1, \cdots, k\} \text{ with } I \neq J \tag{10}$$

providing (1) forms an exact m-cover of $\mathbb{Z}$.

In the present paper we continue our investigation on the basis of [21]. Apart from the last section which contains some unsolved problems arising naturally, all the remaining sections will be devoted to further properties of the common differences in an (exact) m-cover of $\mathbb{Z}$ which are closely connected with Egyptian fractions and some known results introduced above. Namely, when (1) forms an (exact) m-cover of $\mathbb{Z}$ our theorems tell information about the rationals $\Sigma_{s\in I} m_s/n_s, I \subseteq \{1, \cdots, k\}$ and their fractional parts where $m_1, \cdots, m_k$ are suitable positive integers. For the sake of clarity we give below two collections of our central results in the simplest case while we actually prove more.

Theorem I. *Let (1) be an m-cover of $\mathbb{Z}$ with $m \in \mathbb{Z}^+$. Then*

(i) For any $m_1, \cdots, m_k \in \mathbb{Z}^+$ there exist at least m positive integers representable by $\Sigma_{s\in I} m_s/n_s$ with $I \subseteq \{1, \cdots, k\}$.

(ii) If $m > 1$, then for any $m_1, \cdots, m_k \in \mathbb{Z}^+$ and $t = 1, \cdots, k$

$$\sum_{s\in I} \frac{m_s}{n_s} \in \mathbb{Z}^+ \qquad \textit{for some } I \subseteq \{1, \cdots, k\} \textit{ with } t \notin I. \tag{11}$$

If $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ n_s \mid n}}^k$ doesn't form an m-cover of $\mathbb{Z}$ with $n \in \mathbb{Z}^+$, then

$$\sum_{s\in I} \frac{(n, n_s)}{n_s} \in \mathbb{Z}^+ \quad \textit{for some } I \subseteq \{1 \leqslant s \leqslant k : \ n_s \nmid n\}.$$

(iii) When $a_t + n_t\mathbb{Z}$ is essential (i.e., $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^k$ fails to be an m-cover of $\mathbb{Z}$), for every $r = 0, 1, \cdots, n_t - 1$

$$\frac{r}{n_t} \equiv \sum_{s\in I_1} \frac{1}{n_s} - \sum_{s\in I_2} \frac{1}{n_s} \pmod 1 \tag{12}$$

for some $I_1, I_2 \subseteq \{1, \cdots, k\} \setminus \{t\}$ with $\Sigma_{s\in I_1} 1/n_s \geqslant m - 1$ and $\Sigma_{s\in I_2} 1/n_s \geqslant m - 2$, and the rationals $\Sigma_{s\in I} 1/n_s$, $I \subseteq \{1, \cdots, k\} \setminus \{t\}$ have at least n_t distinct fractional parts. (For $r_1, r_2 \in \mathbb{Q}$ we write $r_1 \equiv r_2 \pmod 1$ to mean $r_1 - r_2 \in \mathbb{Z}$.)

(iv) Assume (7) with $0 < l \leqslant k$. If $l \neq k$ then

$$l \geqslant \frac{n_k}{n_{k-l}} \quad \text{or} \quad \sum_{s=1}^{k-l} \frac{1}{n_s} \geqslant m. \tag{13}$$

If $n_k \neq 1$, then either there are at least m positive integers in the form $\Sigma_{s\in I} 1/n_s - 1/n_k$ where $I \subseteq \{1, \cdots, k\}$, or l is the sum of some (not necessarily distinct) denominators greater than 1 of the rationals $\Sigma_{s\in I} 1/n_s - 1/n_k$, $I \subseteq \{1, \cdots, k\}$ and therefore not less than $p([n_1, \cdots, n_k])$. (For a rational $r = a/b$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$ and $(a, b) = 1$, b is called its denominator.)

Theorem II. *Let (1) be an exact m-cover of $\mathbb{Z}$ with $m \in \mathbb{Z}^+$. Then*

(i) Let n be a positive integer and v a rational such that there exists a unique $J \subseteq \{1, \cdots, k\}$ with $\Sigma_{s\in J}(n, n_s)/n_s = v$ (e.g. $v = 0$), then for every $t = 1, \cdots, k$ there is an $I \subseteq \{1, \cdots, k\}$ with $t \in I$ such that

$$\sum_{s\in I} \frac{(n, n_s)}{n_s} \equiv v \pmod{1}.$$

(ii) Provided (8) with $0 < l \leqslant k$, $n_s \mid n_k$ for all $s = 1, \cdots, k$ and for each $r \in \mathbb{Z}$ there exists an $I \subseteq \{1, \cdots, k-1\}$ with the property

$$\sum_{s\in I} \frac{n_k}{n_s} \equiv r \pmod{n_k}. \tag{14}$$

(iii) When $m = 1$, for all $t = 1, \cdots, k$ and $r = 0, 1, \cdots, n_t - 1$ we have

$$\frac{r}{n_t} = \sum_{s\in I} \frac{1}{n_s} \quad \text{for some } I \subseteq \{1, \cdots, k\} \text{ with } t \notin I. \tag{15}$$

(iv) Assume (7) with $0 < l < k$, then for any positive integer $\lambda < n_k/n_{k-l} \leqslant l$, $\binom{l}{\lambda}$ can be written as the sum of some denominators greater than 1 of the rationals

$$\sum_{s\in I} \frac{1}{n_s} - \frac{\lambda}{n_k}, \qquad I \subseteq \{1, \cdots, k\}.$$

2. Connection between Covers of $\mathbb{Z}$ and Covers of $a + n\mathbb{Z}$

From now on we let $a_1, \cdots, a_k$ be integers and $m, n, n_1, \cdots, n_k$ positive ones unless they are specified.

Clearly if (1) is an (exact) m-cover of $\mathbb{Z}$—the only $AS(1)$, it is also an (exact) m-cover of any $AS(n)$. As for the converse we have

Lemma 1. *Let a be an integer. Then $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ forms an (exact) m-cover of $a + n\mathbb{Z}$ if and only if $A' = \{q_j + (n_j/(n, n_j))\mathbb{Z}\}_{j\in J}$ forms an (exact) m-cover of $\mathbb{Z}$ where for each $j \in J = \{1 \leqslant s \leqslant k : (n, n_s) \mid a_s - a\}$, q_j is such an integer that $a + q_j n \equiv a_j \pmod{n_j}$.*

Proof. If $1 \leqslant s \leqslant k$ and $s \notin J$, then $a_s + n_s\mathbb{Z} \cap a + n\mathbb{Z} = \emptyset$. If $j \in J$ and $x \in \mathbb{Z}$, then

$$\begin{aligned} x \in q_j + \frac{n_j}{(n, n_j)}\mathbb{Z} &\iff \frac{n_j}{(n, n_j)} \,\Big|\, \frac{n}{(n, n_j)}(x - q_j) \\ &\iff a + nx \equiv a + nq_j \equiv a_j \pmod{n_j}. \end{aligned}$$

So the desired result follows.

Corollary 1. *Provided (1) is an exact m-cover of $\mathbb{Z}$ we have*

(i) For every $v = 0, 1, \cdots, m-1$

$$\sum_{s\in I}\frac{(n,n_s)}{n_s} = v \quad \textit{for at least } \binom{m}{v} \textit{ subsets } I \textit{ of } \{1,\cdots,k\}. \tag{16}$$

(ii) For each $t = 1, \cdots, k$,

$$\sum_{s\in I}\frac{(n,n_s)}{n_s} = m \quad \textit{for some } I \subseteq \{1,\cdots,k\} \textit{ with } t \in I. \tag{17}$$

(iii) Let $\emptyset \subset J \subset \{1, \cdots, k\}$. Then either for all $j \in J$ and $s \notin J$ we have $(n, n_j, n_s) \nmid a_j - a_s$ and hence $(n, n_j, n_s) > 1$; or

$$\sum_{s\in I}\frac{(n,n_s)}{n_s} = \sum_{s\in J}\frac{(n,n_s)}{n_s} \quad \textit{for some } I \subseteq \{1,\cdots,k\} \textit{ with } I \neq J. \tag{18}$$

Proof. a) Let $1 \leqslant t \leqslant k$. Clearly $t \in I(t) = \{1 \leqslant s \leqslant k : \ (n, n_s) \mid a_s - a_t\}$. For each $s \in I(t)$ we define q_{st} to be an integer such that

$$a_t + q_{st}n \equiv a_s \pmod{n}.$$

By Lemma 1 system $A_t = \{q_{st} + (n_s/(n, n_s))\mathbb{Z}\}_{s\in I(t)}$ forms an exact m-cover of $\mathbb{Z}$, therefore $\Sigma_{s\in I(t)}(n, n_s)/n_s = m$ which proves part (ii). Applying the theorem of [20] to A_t we obtain (i).

b) Let $\emptyset \neq J \subset \{1, \cdots, k\}$ and assume that $(n, n_j, n_t) \mid a_j - a_t$ for some $j \in J$ and $t \notin J$. Put

$$x \in a_j + (n, n_j)\mathbb{Z} \cap a_t + (n, n_t)\mathbb{Z} \ \text{ and } \ I = \{1 \leqslant s \leqslant k : \ (n, n_s) \mid x - a_s\}.$$

Obviously $j \in I \cap J$ and $t \in I \setminus J$. For $s \in I$ we let x_s be such an integer that $x + nx_s \equiv a_s \pmod{n_s}$. By Lemma 1 $\{x_s + (n_s/(n, n_s))\mathbb{Z}\}_{s\in I}$ is an exact m-cover of $\mathbb{Z}$. As $\emptyset \neq I \cap J \neq I$ it follows from Theorem 4 of [21] that there exists an $I' \subseteq I$ with $I' \neq I \cap J$ and $\Sigma_{s\in I'}(n_s/(n, n_s))^{-1} = \Sigma_{s\in I\cap J}(n_s/(n, n_s))^{-1}$. Clearly $J' = I' \cup (J \setminus I) \neq J$ and

$$\sum_{s\in J'}\frac{(n,n_s)}{n_s} = \sum_{s\in I\cap J}\frac{(n,n_s)}{n_s} + \sum_{s\in J\setminus I}\frac{(n,n_s)}{n_s} = \sum_{s\in J}\frac{(n,n_s)}{n_s}.$$

We are done.

The following lemma serves as another reason why we sometimes formulate our results in terms of m-covers of a general arithmetic sequence.

Lemma 2. *Let (1) be an (exact) m-cover of $\mathbb{Z}$, and J be a nonempty subset of $\{1, \cdots, k\}$ such that $a + n\mathbb{Z}$ is covered by $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\notin J}}^{k}$ exactly v times where $a, v \in \mathbb{Z}$ and $0 \leqslant v < m$. Then $\{a_j + n_j\mathbb{Z}\}_{j\in J}$ forms an (exact) $(m - v)$-cover of $a + (n, [n_j]_{j\in J}, [n_s]_{s\notin J})\mathbb{Z}$. ($[n_i]_{i\in\emptyset}$ is regarded as 1.)*

Proof. The case $J = \{1, \cdots, k\}$ is trivial, so we assume $\emptyset \subset J \subset \{1, \cdots, k\}$. Since $a + (n, [n_s]_{s\notin J})\mathbb{Z} = a + n\mathbb{Z} + [n_s]_{s\notin J}\mathbb{Z}$ is covered by $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\notin J}}^{k}$ exactly v times,

$$a + (n, [n_j]_{j\in J}, [n_s]_{s\notin J})\mathbb{Z} = a + (n, [n_s]_{s\notin J})\mathbb{Z} + [n_j]_{j\in J}\mathbb{Z}$$

must be covered by $\{a_j + n_j\mathbb{Z}\}_{j\in J}$ at least (resp. exactly) $m - v$ times. The proof is ended.

Corollary 2. *Provided that (1) is an exact m-cover of $\mathbb{Z}$, for every $v = 0, 1, \cdots, m$ and $t = 1, \cdots, k$ with $(n, n_t) > 1$,*

$$\sum_{s\in I}\frac{(n,n_s)}{n_s} = v \quad \textit{for at least } \binom{m}{v} \textit{ subsets } I \textit{ of } \{1,\cdots,k\}\setminus\{t\}. \tag{19}$$

Proof. Let $0 \leqslant v \leqslant m$, $1 \leqslant t \leqslant k$ and $(n, n_t) > 1$. Clearly $1+a_t+n\mathbb{Z}\cap a_t+n_t\mathbb{Z} = \emptyset$. By Lemma 2 $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^{k}$ forms an exact m-cover of $1 + a_t + (n, [n_s]_{\substack{s=1\\ s\neq t}}^{k})\mathbb{Z} \subseteq 1 + a_t + (n, [n_s]_{\substack{s=1\\ s\neq t}}^{k}, n_t)\mathbb{Z}$. Now the desired (19) follows from Lemma 1 and the theorem of [20].

Corollary 3. *Suppose that (1) is an m-cover of $\mathbb{Z}$ but it won't be if we omit all the $a_j + n_j\mathbb{Z}$, $j \in J$ from (1) where $\emptyset \neq J \subseteq \{1, \cdots, k\}$. Then*

$$|J| \geqslant \min_{j\in J}\frac{n_j}{(n_j,[n_s]_{s\notin J})}, \tag{20}$$

that is, there exists a $j \in J$ such that $[n_j, [n_s]_{s\notin J}] \leqslant |J|[n_s]_{s\notin J}$.

Proof. Let a be an integer with

$$v = |\{1 \leqslant s \leqslant k :\ s \notin J \text{ and } a \in a_s + n_s\mathbb{Z}\}| < m.$$

Then $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\notin J}}^{k}$ forms an exact v-cover of $a + [n_s]_{s\notin J}\mathbb{Z}$. By Lemma 2, system $\{a_j + n_j\mathbb{Z}\}_{j\in J}$ is a cover of $a + ([n_j]_{j\in J}, [n_s]_{s\notin J})\mathbb{Z}$. In view of Lemma 1

$$\sum_{j\in J}\frac{(n_j,[n_s]_{s\notin J})}{n_j} \geqslant \sum_{\substack{j\in J\\ (n_j,[n_s]_{s\notin J})\mid a_j-a}}\frac{1}{n_j/(n_j,[n_s]_{s\notin J})} \geqslant 1$$

and in fact by Zhang's result ([25])

$$\sum_{j\in I}\frac{1}{n_j/(n_j,[n_s]_{s\notin J})} \in \mathbb{Z}^+ \quad \text{for some } I \subseteq \{j \in J :\ (n_j, [n_s]_{s\notin J}) \mid a_j - a\}.$$

Since $\Sigma_{j\in J}(n_j, [n_s]_{s\notin J})/n_j \not< 1 = \Sigma_{j\in J}1/|J|$, there exists a $j \in J$ such that

$$\frac{n_j}{(n_j,[n_s]_{s\notin J})} = \frac{[n_j,[n_s]_{s\notin J}]}{[n_s]_{s\notin J}} \leqslant |J|.$$

This concludes the proof.

Note that when $J = \{1 \leqslant s \leqslant k :\ d \mid n_s\}$ with $d > n_0 = 1$ (20) yields (2).

3. Lower Bound for the Cardinality of $\{\{\Sigma_{s\in I}m_s/n_s\} :\ I \subseteq \{1, \cdots, k\}\}$

In 1970 R. B. Crittenden and C. L. Vanden Eynden ([5]) proved a conjecture of Erdös ([7]) with prizes which states that if (1) doesn't form a cover of $\mathbb{Z}$ then there exists an integer x with $1 \leqslant x \leqslant 2^k$ which is not covered by (1). In [21] we obtained the following stronger result.

Lemma 3. *Let $\alpha_1, \cdots, \alpha_k$ be reals and $\beta_1, \cdots, \beta_k$ positive reals. Then system $\{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^{k}$ forms an m-cover of $\mathbb{Z}$ if it covers $|\{\{\Sigma_{s\in I}1/\beta_s\} :\ I \subseteq \{1, \cdots, k\}\}|$ ($\leqslant 2^k$) consecutive integers at least m times. (As usual $\{x\}$ and $[x] = x - \{x\}$ denote the fractional and the integral parts of a real x respectively.)*

An ingenious application of Lemma 3 gives

Theorem 1. *Let a be an integer and $m_1, \cdots, m_k$ positive integers with $(m_s, n_s) = (n, n_s)$ for every $s = 1, \cdots, k$.*

(i) If $A = a_s + n_s\mathbb{Z}$ covers

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I \subseteq \{1 \leqslant s \leqslant k:\ (n, n_s) \mid a_s - a\} \subseteq \{1, \cdots, k\}\right\}\right|$$

consecutive integers congruent to a modulo n at least m times, then A forms an m-cover of $a + n\mathbb{Z}$.

(ii) If for some divisor d of n (1) is an m-cover of $a+d\mathbb{Z}$ but $A_t = \{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^k$ is not, then

$$\begin{aligned}&\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I \subseteq \{1, \cdots, k\}\setminus\{t\}\right\}\right|\\ &\quad\geqslant \left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I \subseteq \{1 \leqslant s \leqslant k:\ s \neq t\ \&(d, n_s) \mid a_s - a\}\right\}\right|\\ &\quad\geqslant \frac{n_t}{(n, n_t)}.\end{aligned} \tag{21}$$

Proof. i) Let $J = \{1 \leqslant s \leqslant k:\ (n, n_s) \mid a_s - a\}$ and define $q_j,\ j \in J$ as in Lemma 1. For $j \in J$ and $x \in \mathbb{Z}$ we have

$$\begin{aligned}a + nx \in a_j + n_j\mathbb{Z} &\Longleftrightarrow x \in q_j + \frac{n_j}{(n, n_j)}\mathbb{Z} = q_j + \frac{n_j}{(m_j, n_j)}\mathbb{Z}\\ &\Longleftrightarrow x \in q_j + \frac{n_j/(m_j, n_j)}{m_j/(m_j, n_j)}\mathbb{Z} = q_j + \frac{n_j}{m_j}\mathbb{Z}.\end{aligned}$$

As (1) covers $|\{\{\Sigma_{s\in I}m_s/n_s\} : I \subseteq J\}|$ consecutive terms in $a+n\mathbb{Z}$ at least m times, $B = \{q_j+(n_j/m_j)\mathbb{Z}\}_{j\in J}$ forms an m-cover of some $AP_1(|\{\{\Sigma_{s\in I}m_s/n_s\} :\ I \subseteq J\}|)$. By Lemma 3 B is an m-cover of $\mathbb{Z}$, so A is an m-cover of $a + n\mathbb{Z}$.

ii) Suppose that d is a divisor of n for which $A = \{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms an m-cover of $a + d\mathbb{Z}$ with $a_t + n_t\mathbb{Z}$ essential. Since $a + d\mathbb{Z}$ is the union of $a + rd + n\mathbb{Z},\ r = 0, 1, \cdots, n/d - 1$, for some $r = 0, 1, \cdots, n/d - 1$, $A_t = \{a_s + n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^k$ doesn't form an m-cover of $a + rd + n\mathbb{Z}$ but A does. Clearly $a + rd + n\mathbb{Z} \cap a_t + n_t\mathbb{Z}$ contains some integer x. Note that

$$x + n,\ x + 2n,\ \cdots,\ x + ([n, n_t]/n - 1)n$$

are $[n, n_t]/n - 1 = n_t/(n, n_t) - 1$ consecutive terms in $a + rd + n\mathbb{Z}$ each of which is covered by A_t at least m times because they don't belong to $a_t + n_t\mathbb{Z}$. By part (i)

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I \subseteq \{1 \leqslant s \leqslant k:\ s \neq t\ \&\ (n, n_s) \mid a_s - (a + rd)\}\right\}\right| \nleqslant \frac{n_t}{(n, n_t)} - 1,$$

therefore

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1\leqslant s\leqslant k:\ s\neq t\ \&\ (d,n_s)\mid a_s-a\}\right\}\right|$$
$$\geqslant\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1\leqslant s\leqslant k:\ s\neq t\ \&\ (n,n_s)\mid a_s-(a+rd)\}\right\}\right|\geqslant\frac{n_t}{(n,n_t)}$$

which completes the proof.

Corollary 4. *(i)* $\bigcup_{s=1}^k a_s+n_s\mathbb{Z}$ *contains an* $AP_n(l)$ *for each* $l=1,2,3,\cdots$ *if and only if it contains an* $AP_n(\bar{n})$ *where* $\bar{n}$ *is the least cardinality of those sets* $\{\{\Sigma_{s\in I}m_s/n_s\}:\ I\subseteq\{1,\cdots,k\}\}$ *with* $m_s\in\mathbb{Z}^+$ *and* $(m_s,n_s)=(n,n_s)$ *for all* $s=1,\cdots,k$.

(ii) Providing $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ *is a minimal* m*-cover of* $\mathbb{Z}$ *with at least* n_0 *distinct numbers among* $n_1,\cdots,n_k$ *where* n_0 *is an explicitly computable constant which can be* 10^{70} *under the Riemann hypothesis, we have*

$$\max_{1\leqslant t\leqslant k}\left|\left\{\left\{\sum_{s\in I}c_s\frac{(n_s,n_t)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\geqslant|\{n_1,\cdots,n_k\}| \tag{22}$$

where $c_1,\cdots,c_k$ *are any integers prime to* $n_1,\cdots,n_k$ *respectively.*

Proof. i) Part (i) follows immediately from the first part of Theorem 1 in the case $m=1$.

ii) In 1970 R. L. Graham conjectured that for any l distinct positive integers $x_1,\cdots,x_l$ one has the inequality $\max_{1\leqslant i,j\leqslant l}x_i/(x_i,x_j)\geqslant l$. M. Szegedy [24] proved this for $l\geqslant n_0$ where n_0 is an explicitly computable constant. According to [4], under the Riemann hypothesis we can take $n_0=10^{70}$.

Set $S=\{n_1,\cdots,n_k\}$. As $|S|\geqslant n_0$, by Szegedy's result

$$\max_{1\leqslant i,j\leqslant k}\frac{n_i}{(n_i,n_j)}=\max_{a,b\in S}\frac{a}{(a,b)}\geqslant|S|$$

and hence

$$\frac{n_j}{(n_j,n_t)}\geqslant|\{n_1,\cdots,n_k\}|\qquad\text{for some } j,t=1,\cdots,k.$$

Since $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ is an m-cover of $\mathbb{Z}=0+1\mathbb{Z}$ but $A_j=\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq j}}^k$ is not, it follows from part (ii) of Theorem 1 that

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\setminus\{j\}\right\}\right|\geqslant\frac{n_j}{(n_j,n_t)}$$

where each m_s is a positive integer congruent to $c_s(n_s,n_t)$ modulo n_s. Therefore

$$\left|\left\{\left\{\sum_{s\in I}c_s\frac{(n_s,n_t)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\geqslant\frac{n_j}{(n_j,n_t)}\geqslant|\{n_1,\cdots,n_k\}|.$$

This proves part (ii).

Corollary 5. *Let* $c_1,\cdots,c_k$ *be integers with* $(c_s,n_s/(n,n_s))=1$ *for all* $s=1,\cdots,k-1$.

(i) Suppose that $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ is an m-cover of some $AS(d)$ with $a_k+n_k\mathbb{Z}$ essential where $d\mid n$ and $n_s\mid[n,n_k]$ for all $s=1,\cdots,k-1$. Then

$$\left\{\left\{\sum_{s\in I}c_s\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}$$
$$=\left\{\left\{\sum_{s\in I}c_s\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k-1\}\right\} \tag{23}$$
$$=\left\{\frac{(n,n_k)}{n_k}r:\ r=0,1,\cdots,\frac{n_k}{(n,n_k)}-1\right\},$$

that is, $\{\Sigma_{s\in I}c_s[n,n_k]/[n,n_s]:\ I\subseteq\{1,\cdots,k-1\}\}$ contains a complete system of residues modulo $[n,n_k]/n$.

(ii) (23) holds if (1) forms an exact m-cover of $\mathbb{Z}$ with (8) where $0<l\leqslant k$.

Proof. i) For every $s=1,\cdots,k$, $n_s\mid nn_k/(n,n_k)$ and hence $n_s/(n,n_s)$ divides $n_k/(n,n_k)$. Since

$$\left\{\sum_{s\in I}c_s\frac{(n,n_s)}{n_s}:\ I\subseteq\{1,\cdots,k\}\right\}\subseteq\left\{x\Big/\left[\frac{n_1}{(n,n_1)},\cdots,\frac{n_k}{(n,n_k)}\right]:\ x\in\mathbb{Z}\right\},$$

we have

$$\left|\left\{\left\{\sum_{s\in I}c_s\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\leqslant\left[\frac{n_1}{(n,n_1)},\cdots,\frac{n_k}{(n,n_k)}\right]=\frac{n_k}{(n,n_k)}.$$

On the other hand, by Theorem 1

$$\left|\left\{\left\{\sum_{s\in I}c_s\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq\{1,\cdots,k-1\}\right\}\right|\geqslant\frac{n_k}{(n,n_k)}.$$

So (23) must be true.

ii) Let (1) be an exact m-cover of $\mathbb{Z}$ with (8) where $0<l\leqslant k$. Then

$$|\{1\leqslant s\leqslant k:\ x\in a_s+n_s\mathbb{Z}\}|=m\quad\text{for any } x\in\mathbb{Z}.$$

We claim that $n_s\mid n_k$ for all $s=1,\cdots,k$.

Apparently $n_s\mid n_k$ for $s>k-l$. Let $1\leqslant j\leqslant k-l$ and assume that $n_s\mid n_k$ for those $s>j$. When $n_j>n_0=1$ it follows from Theorem 1 of [19] that (6) holds for $d=n_j$. So $n_j\mid n_t$ for some $t>j$. Since $n_t\mid n_k$ by the assumption, n_j does divide n_k. This proves the claim by induction.

Applying part (i) to (1), a minimal m-cover of $\mathbb{Z}=AS(1)$, we then obtain the desired (23).

Remark. Clearly (23) implies that $n_s/(n,n_s)$ divides $n_k/(n,n_k)$ (i.e. $n_s\mid[n,n_k]$) for all $s=1,\cdots,k$. When $c_1=\cdots=c_k=n=1$, part (i) of Corollary 5 in the case $m=1$ is noted by the author's brother Zhi-Hong Sun who has learned Lemma 3 from [21], and part (ii) of Corollary 5 is equivalent to the second part of Theorem II.

Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be a minimal m-cover of $\mathbb{Z}$. By Theorem 1 for each $t=1,\cdots,k$ we have

$$n_t\leqslant\left|\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\setminus\{t\}\right\}\right|\leqslant\min\{[n_s]_{\substack{s=1\\ s\neq t}}^k,\ 2^{k-1}\}$$

where the upper bound is evident. Putting $J=\{t\}$ in (20) we get that $n_t \mid [n_s]_{\substack{s=1\\ s\neq t}}^k$, thus $N=[n_1,\cdots,n_k]$ equals $[n_s]_{\substack{s=1\\ s\neq t}}^k$. Letting $d=1$ in (3) we obtain that $k\geqslant 1+f(N)$, therefore $N\leqslant 2^{f(N)}\leqslant 2^{k-1}$. So the above upper bound coincides with N. For the set

$$S(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\} \tag{24}$$

we have

$$\max_{1\leqslant s\leqslant k} n_s\leqslant |S(A)|\leqslant N=[n_1,\cdots,n_k]. \tag{25}$$

Recently Z. H. Sun conjectured that if $m=1$ then

$$|S(A)|=N,\quad \text{i.e.}\quad S(A)=\left\{0,\frac{1}{N},\cdots,\frac{N-1}{N}\right\}.$$

The following examples refute the conjecture.

Example 1. $A_1=\{2\mathbb{Z},3+8\mathbb{Z},7+8\mathbb{Z},1+12\mathbb{Z},5+12\mathbb{Z},9+12\mathbb{Z}\}$ is an exact cover of $\mathbb{Z}$ with

$$n_1=2<n_2=n_3=8<n_4=n_5=n_6=12\quad\text{and}\quad N=[n_1,\cdots,n_6]=24.$$

It is easy to verify that $|S(A_1)|=20$, in fact

$$S(A_1)=\left\{\frac{r}{24}:\ r\in\mathbb{Z},\ 0\leqslant r<24 \text{ and } r\neq 1,11,13,23\right\}$$

Example 2. $A_2=\{1+4\mathbb{Z},3+4\mathbb{Z},6\mathbb{Z},2+6\mathbb{Z},4+6\mathbb{Z}\}$ forms an exact cover of $\mathbb{Z}$ with $n_1=n_2=4<n_3=n_4=n_5=6$. For $s=1,2,3,4,5$, $\{1,-1\}$ contains a reduced set of residues modulo $n_s/(n,n_s)$ since $\varphi(n_s/(n,n_s))\leqslant 2$ where φ is the Euler totient function. Set

$$M_n=\max_{1\leqslant s\leqslant 5}\frac{n_s}{(n,n_s)},\quad N_n=\left[\frac{n_1}{(n,n_1)},\cdots,\frac{n_5}{(n,n_5)}\right]$$

and

$$S_n(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)=\left\{\left\{\sum_{s\in I}\varepsilon_s\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq\{1,\cdots,5\}\right\}\text{ for }\varepsilon_1,\cdots,\varepsilon_5=\pm1.$$

It is easy to check that when $2\mid n$ or $3\mid n$ we have

$$S_n(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)=\left\{0,\frac{1}{N_n},\cdots,\frac{N_n-1}{N_n}\right\}\text{ for all }\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\in\{1,-1\}.$$

If $(6,n)=1$, then $M_n=6,\ N_n=12$, and by trivial computations

$$\begin{aligned}
&S_n(1,1,1,1,1)=S_n(-1,-1,-1,-1,-1)=\{1,11\}_{12}^-,\\
&S_n(1,-1,1,1,1)=S_n(-1,1,1,1,1)=\{8,10\}_{12}^-,\\
&S_n(1,-1,-1,-1,-1)=S_n(-1,1,-1,-1,-1)=\{2,4\}_{12}^-,\\
&S_n(1,1,-1,-1,-1)=S_n(-1,-1,1,1,1)=\{5,7\}_{12}^-;\\
&S_n(1,1,-1,1,1)=S_n(1,1,1,-1,1)=S_n(1,1,1,1,-1)=\{9,11\}_{12}^-,\\
&S_n(-1,-1,1,-1,-1)=S_n(-1,-1,-1,1,-1)=S_n(-1,-1,-1,-1,1)=\{1,3\}_{12}^-,\\
&S_n(1,1,-1,-1,1)=S_n(1,1,-1,1,-1)=S_n(1,1,1,-1,-1)=\{7,9\}_{12}^-,\\
&S_n(-1,-1,1,1,-1)=S_n(-1,-1,1,-1,1)=S_n(-1,-1,-1,1,1)=\{3,5\}_{12}^-;\\
&S_n(1,-1,1,1,-1)=S_n(1,-1,-1,1,1)=S_n(1,-1,1,-1,1)\\
&\ =S_n(-1,1,1,1,-1)=S_n(-1,1,-1,1,1)=S_n(-1,1,1,-1,1)=\{6,7,8\}_{12}^-,\\
&S_n(-1,1,-1,-1,1)=S_n(-1,1,1,-1,-1)=S_n(-1,1,-1,1,-1)\\
&\ =S_n(1,-1,-1,-1,1)=S_n(1,-1,1,-1,-1)=S_n(1,-1,-1,1,-1)=\{4,5,6\}_{12}^-
\end{aligned}$$

where for $X\subseteq\{0,1,\cdots,11\}$ we use X_{12}^- to denote the set

$$\left\{\frac{r}{12}:\ r\in\mathbb{Z},\ 0\leqslant r<12\text{ and }r\notin X\right\}.$$

By part (i) of Theorem 1 or Corollary 4, when

$$A_3=\{a_1+4\mathbb{Z},\ a_2+4\mathbb{Z},\ a_3+6\mathbb{Z},\ a_4+6\mathbb{Z},\ a_5+6\mathbb{Z}\}$$

covers 0,1,2,3,4,5,6,7,8 it forms a cover of $\mathbb{Z}$. Actually one can prove that A_3 is a cover of $\mathbb{Z}$ if it covers integers from 0 to 7. On the other hand,

$$A_4=\{1+4\mathbb{Z},\ 2+4\mathbb{Z},\ 6\mathbb{Z},\ 3+6\mathbb{Z},\ 4+6\mathbb{Z}\}$$

covers 0,1,2,3,4,5,6 but it doesn't cover all the integers.

In contrast with part (ii) of Corollary 5 we note that when $(6,n)=c_1=1$ and $c_2=c_3=c_4=c_5=-1$, $(n,n_5)/n_5$ is not the fractional part of $\Sigma_{s\in I}c_s(n,n_s)/n_s$ for any $I\subseteq\{1,2,3,4,5\}$ because $1/6\notin S_n(1,-1,-1,-1,-1)$.

4. Existence of Distinct $I_1,I_2\subseteq\{1,\cdots,k\}$ with $\{\sum_{s\in I_1}\frac{m_s}{n_s}\}=\{\sum_{s\in I_2}\frac{m_s}{n_s}\}$

In [21] we established the following result.

Lemma 4. *Let $\alpha_1,\cdots,\alpha_k$ be reals and $\beta_1,\cdots,\beta_k$ positive reals. Then the following conditions are equivalent.*

(a) $\{\alpha_s+\beta_s\mathbb{Z}\}_{s=1}^k$ forms an m-cover of $\mathbb{Z}$.

(b) For any $\theta\in\{\{\Sigma_{s\in I}1/\beta_s\}:\ I\subseteq\{1,\cdots,k\}\}$ and $l=0,1,\cdots,m-1$,

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\Sigma_{s\in I}1/\beta_s]}{l}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}=0.$$

An easy consequence of this lemma is the following extension of Zhang's result ([25]): If (1) is an m-cover of $\mathbb{Z}$ then for any $I_1\subseteq\{1,\cdots,k\}$ there exists an $I_2\subseteq\{1,\cdots,k\}$ different from I_1 such that $\{\Sigma_{s\in I_1}1/n_s\}=\{\Sigma_{s\in I_2}1/n_s\}$. (See Theorem

2 of [21] and compare this with the one related to (10).) In the present section we will strengthen this result.

Lemma 5. *Let $a \in \mathbb{Z}$, $J = \{1 \leqslant s \leqslant k :\ (n, n_s) \mid a_s - a\}$ and $m_j \in \mathbb{Z}^+$ for each $j \in J$. Then (i) implies (ii) if $(n, n_j) \mid m_j$ for all $j \in J$, and (ii) implies (i) if $(m_j, n_j) = (n, n_j)$ for every $j \in J$ where statements (i) and (ii) are as follows:*

(i) $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ covers $|S|$ consecutive terms in $a + n\mathbb{Z}$ at least m times where

$$S = \left\{0 \leqslant \theta < 1 :\ \left\{\sum_{s\in I} \frac{m_s}{n_s}\right\} = \theta \text{ for some } I \subseteq J\right\}. \tag{26}$$

(ii) For each $\theta \in S$ and $l = 0, 1, \cdots, m-1$,

$$\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I} m_s/n_s\}=\theta}} (-1)^{|I|} \binom{[\Sigma_{s\in I} m_s/n_s]}{l} e^{2\pi i \Sigma_{s\in I} q_s m_s/n_s} = 0 \tag{27}$$

where $q_j,\ j \in J$ are integers such that $a + nq_j \equiv a_j \pmod{n_j}$ for all $j \in J$.

Proof. Clearly (i) holds if and only if $B = \{q_j + (n_j/(n, n_j))\mathbb{Z}\}_{j\in J}$ forms an m-cover of some $AP_1(|S|)$ (see the proof of Lemma 1). On the other hand, by Lemmas 3 and 4,

$$\begin{aligned}
&\text{(ii) is valid}\\
\Longleftrightarrow&\{q_j + (n_j/m_j)\mathbb{Z}\}_{j\in J} \text{ is an } m\text{-cover of } \mathbb{Z}\\
\Longleftrightarrow&\left\{q_j + \frac{n_j/(m_j, n_j)}{m_j/(m_j, n_j)}\mathbb{Z}\right\}_{j\in J} \text{ covers } |S| \text{ consecutive integers at least } m \text{ times}\\
\Longleftrightarrow&\{q_j + (n_j/(m_j, n_j))\mathbb{Z}\}_{j\in J} \text{ forms an } m\text{-cover of some } AP_1(|S|).
\end{aligned}$$

We are done because for each $j \in J$, $q_j + (n_j/(n, n_j))\mathbb{Z} \subseteq q_j + (n_j/(m_j, n_j))\mathbb{Z}$ if $(n, n_j) \mid m_j$, and the equality holds if $(m_j, n_j) = (n, n_j)$.

Theorem 2. *Let m_* be a nonnegative integer less than m, and $m_1, \cdots, m_k$ be positive integers divisible by $(n, n_1), \cdots, (n, n_k)$ respectively. Suppose that for some integer a and divisor d of n there are*

$$\frac{n}{d}\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} :\ I \subseteq \{1 \leqslant s \leqslant k :\ (d, n_s) \mid a_s - a\}\right\}\right|$$

consecutive terms in $a + d\mathbb{Z}$ covered by $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ and $B = \{a_s + n_s\mathbb{Z}\}_{L\setminus K}$ at least m and m_* times respectively and B is not an m-cover of $a + d\mathbb{Z}$ where K and L are subsets of $\{1, \cdots, k\}$ with $K \subseteq L$ and $(m_s, n_s) \mid n$ for all $s \in L \setminus K$. Then there exist $I_1, I_2 \subseteq \{1, \cdots, k\}$ for which we have

(i) $\{\Sigma_{s\in I_1} m_s/n_s\} = \{\Sigma_{s\in I_2} m_s/n_s\}$, $K \subseteq I_1 \subseteq L$ holds and $K \subseteq I_2 \subseteq L$ fails.

(ii) $[\Sigma_{s\in I_1} m_s/n_s] \geqslant m_* + [\Sigma_{s\in K} m_s/n_s]$, $[\Sigma_{s\in I_2} m_s/n_s] \geqslant m_*$ and moreover

$$\left[\sum_{s\in I_2}\frac{m_s}{n_s}\right] \geqslant \begin{cases} m_* + [\sum_{s\in K}\frac{m_s}{n_s}], & \text{if } [\sum_{s\in I_1}\frac{m_s}{n_s}] \leqslant m-1;\\ m-1, & \text{if } [\sum_{s\in I_1}\frac{m_s}{n_s}] = m_* + [\sum_{s\in K}\frac{m_s}{n_s}] \geqslant m-1.\end{cases}$$

Proof. Since $a+d\mathbb{Z}=\bigcup_{b\equiv a \pmod{d}} b+n\mathbb{Z}$, for some $b\equiv a \pmod{d}$ B doesn't form an m-cover of $b+n\mathbb{Z}$ but A and B cover $|S|$ consecutive terms in $b+n\mathbb{Z}$ at least m and m_* times respectively where S is given by (26) with

$$J=\{1\leqslant s\leqslant k:\ (n,n_s)\mid a_s-b\}\subseteq\{1\leqslant s\leqslant k:\ (d,n_s)\mid a_s-a\}.$$

For $j\in J$ we let q_j be such an integer that $b+nq_j\equiv a_j \pmod{n_j}$. Clearly $(m_j,n_j)=(n,n_j)$ for $j\in J\cap(L\setminus K)$. As B is not an m-cover of $b+n\mathbb{Z}$, by Theorem 1 and Lemma 5 for some $I_0\subseteq J\cap(L\setminus K)$ it's impossible that $G(u)=0$ for all $u=0,1,\cdots,m-1$ where

$$G(u)=\sum_{\substack{I\subseteq J\cap(L\setminus K)\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}}(-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{u}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}$$

and $\theta=\{\Sigma_{s\in I_0}m_s/n_s\}$. Since B covers $|\{\{\Sigma_{s\in I}m_s/n_s\}:\ I\subseteq J\cap(L\setminus K)\}|$ consecutive terms in $b+n\mathbb{Z}$ at least m_* times, it follows from Lemma 5 that $G(u)=0$ for every $u=0,1,\cdots,m_*-1$.

Choose u_0 and u_1 to be the maximal and the minimal elements of the set

$$\{u\in\mathbb{Z}:\ 0\leqslant u<m \text{ and } G(u)\neq 0\}$$

respectively. Apparently $m>u_0\geqslant u_1\geqslant m_*$. Now that $G(u_0)\neq 0$, there exists an $I_0'\subseteq J\cap(L\setminus K)$ such that $\{\Sigma_{s\in I_0'}m_s/n_s\}=\theta=\{\Sigma_{s\in I_0}m_s/n_s\}$ and that

$$\binom{[\Sigma_{s\in I_0'}m_s/n_s]}{u_0}\neq 0,\quad\text{i.e.}\quad \left[\sum_{s\in I_0'}\frac{m_s}{n_s}\right]\geqslant u_0.$$

Obviously $J\cap K\subseteq I_1'=I_0'\cup(J\cap K)\subseteq J\cap L$. Set

$$c=\sum_{s\in J\cap K}\frac{m_s}{n_s}+\theta\quad\text{and}\quad u_2=\min\{m-1,\ u_0+[c]\}.$$

For each $l=0,1,2,\cdots$, if $p,q\in\mathbb{Z}^+$ then $\binom{p+q}{l}=\sum_{j=0}^{l}\binom{p}{j}\binom{q}{l-j}$ by the binomial theorem and the identity $(1+z)^{p+q}=(1+z)^p(1+z)^q$, so polynomials $\binom{x+y}{l}$ and $\sum_{j=0}^{l}\binom{x}{j}\binom{y}{l-j}$ are identical, which is known as Vandermonde's identity. Now for

$t=1,2$ we clearly have

$$\begin{aligned}\Sigma_t &= \sum_{\substack{I\subseteq J\cap(L\setminus K)\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s+\Sigma_{s\in J\cap K}m_s/n_s]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&= \sum_{\substack{I\subseteq J\cap(L\setminus K)\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]+[c]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&= \sum_{v=0}^{u_t}\sum_{\substack{I\subseteq J\cap(L\setminus K)\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\binom{[c]}{u_t-v}\binom{[\Sigma_{s\in I}m_s/n_s]}{v}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&= \sum_{\substack{u_1\leqslant v\leqslant u_0\\ u_t-[c]\leqslant v\leqslant u_t}}\binom{[c]}{u_t-v}\sum_{\substack{I\subseteq J\cap(L\setminus K)\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{v}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&= \begin{cases} G(u_1)\neq 0 & \text{if } t=1,\\ G(u_0)\neq 0 & \text{if } t=2 \text{ and } u_0+[c]\leqslant m-1,\\ \binom{[c]}{m-1-u_0}G(u_0)\neq 0 & \text{if } t=2,\ u_0+[c]\geqslant m-1 \text{ and } u_0=u_1.\end{cases}\end{aligned}$$

Observe that each $I\subseteq J\cap L$ with $I\supseteq J\cap K$ and $\{\Sigma_{s\in I}m_s/n_s\}=\{\Sigma_{s\in I_1'}m_s/n_s\}$ can be expressed as the union of $J\cap K$ and a unique $I'\subseteq J\cap(L\setminus K)$ with $\{\Sigma_{s\in I'}m_s/n_s\}=\{\Sigma_{s\in I_0'}m_s/n_s\}$. Therefore, except for the case in which $t=2$ and $u_0>\max\{u_1,m-1-[c]\}$,

$$\begin{aligned}&\sum_{\substack{J\cap K\subseteq I\subseteq J\cap L\\ \{\Sigma_{s\in I}m_s/n_s\}=\{\Sigma_{s\in I_1'}m_s/n_s\}}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&\qquad = (-1)^{|J\cap K|}e^{2\pi i\Sigma_{s\in J\cap K}q_sm_s/n_s}\Sigma_t\neq 0\end{aligned}$$

and thus

$$\begin{aligned}\Sigma_t' &= \sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}=\{\Sigma_{s\in I_1'}m_s/n_s\}\\ J\cap K\subseteq I\subseteq J\cap L\text{ fails}}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&= \sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}=\{\Sigma_{s\in I_1'}m_s/n_s\}}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&\quad - \sum_{\substack{J\cap K\subseteq I\subseteq J\cap L\\ \{\Sigma_{s\in I}m_s/n_s\}=\{\Sigma_{s\in I_1'}m_s/n_s\}}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{u_t}e^{2\pi i\Sigma_{s\in I}q_sm_s/n_s}\\
&\neq 0. \quad \text{(The first sum vanishes by Lemma 5.)}\end{aligned}$$

Let $I_1 = I_1' \cup (K \setminus J)$. Then $K \subseteq I_1 \subseteq (J \cap L) \cup (K \setminus J) \subseteq L$ and

$$\left[\sum_{s\in I_1}\frac{m_s}{n_s}\right] = \left[\sum_{s\in I_0'}\frac{m_s}{n_s} + \sum_{s\in J\cap K}\frac{m_s}{n_s} + \sum_{s\in K\setminus J}\frac{m_s}{n_s}\right] = \left[\sum_{s\in I_0'}\frac{m_s}{n_s}\right] + \left[\theta + \sum_{s\in K}\frac{m_s}{n_s}\right]$$

$$\geqslant u_0 + \left[\theta + \sum_{s\in K}\frac{m_s}{n_s}\right] \geqslant u_1 + \left[\sum_{s\in K}\frac{m_s}{n_s}\right] \geqslant m_* + \left[\sum_{s\in K}\frac{m_s}{n_s}\right].$$

Note that $u_0 = u_1 = m_*$ if $[\Sigma_{s\in I_1} m_s/n_s] = m_* + [\Sigma_{s\in K} m_s/n_s]$.

When $u_0 + [c] \leqslant m-1$ (which is the case if $[\Sigma_{s\in I_1} m_s/n_s] \leqslant m-1$), since $\Sigma_2' \neq 0$ there exists an $I_2' \subseteq J$ without $J \cap K \subseteq I_2' \subseteq J \cap L$ for which

$$\left\{\sum_{s\in I_2'}\frac{m_s}{n_s}\right\} = \left\{\sum_{s\in I_1'}\frac{m_s}{n_s}\right\} = \{c\} \quad\text{and}\quad \left[\sum_{s\in I_2'}\frac{m_s}{n_s}\right] \geqslant u_2 = u_0 + [c],$$

and thus

$$\sum_{s\in I_2'}\frac{m_s}{n_s} \geqslant \{c\} + (u_0 + [c]) = u_0 + c \geqslant m_* + \sum_{s\in J\cap K}\frac{m_s}{n_s}.$$

If $[\Sigma_{s\in I_1} m_s/n_s] = m_* + [\Sigma_{s\in K} m_s/n_s] \geqslant m-1$ and $u_0 + [c] \geqslant m$, then in a similar way there is an $I_2' \subseteq J$ for which

$$\left\{\sum_{s\in I_2'}\frac{m_s}{n_s}\right\} = \left\{\sum_{s\in I_1'}\frac{m_s}{n_s}\right\} \quad\text{and}\quad \left[\sum_{s\in I_2'}\frac{m_s}{n_s}\right] \geqslant u_2 = m-1 \geqslant m_*$$

but $J \cap K \subseteq I_2' \subseteq J \cap L$ fails. In the remaining cases, as $\Sigma_1' \neq 0$ there must exist an $I_2' \subseteq J$ without $J \cap K \subseteq I_2' \subseteq J \cap L$ for which

$$\left\{\sum_{s\in I_2'}\frac{m_s}{n_s}\right\} = \left\{\sum_{s\in I_1'}\frac{m_s}{n_s}\right\} \quad\text{and}\quad \left[\sum_{s\in I_2'}\frac{m_s}{n_s}\right] \geqslant u_1 \geqslant m_*.$$

Now let $I_2 = I_2' \cup (K \setminus J)$. Then $K \subseteq I_2 \subseteq L$ fails, for, otherwise we would have $J \cap K \subseteq J \cap I_2 = I_2' \subseteq J \cap L$ which is impossible. Evidently $\{\Sigma_{s\in I_1} m_s/n_s\} = \{\Sigma_{s\in I_2} m_s/n_s\}$. By the above $[\Sigma_{s\in I_2} m_s/n_s] \geqslant [\Sigma_{s\in I_2'} m_s/n_s] \geqslant m_*$, if $[\Sigma_{s\in I_1} m_s/n_s]$ is less than m then

$$\left[\sum_{s\in I_2}\frac{m_s}{n_s}\right] = \left[\sum_{s\in I_2'}\frac{m_s}{n_s} + \sum_{s\in K\setminus J}\frac{m_s}{n_s}\right]$$

$$\geqslant \left[m_* + \sum_{s\in J\cap K}\frac{m_s}{n_s} + \sum_{s\in K\setminus J}\frac{m_s}{n_s}\right] \geqslant m_* + \left[\sum_{s\in K}\frac{m_s}{n_s}\right];$$

when $[\Sigma_{s\in I_1} m_s/n_s] = m_* + [\Sigma_{s\in K} m_s/n_s] \geqslant m-1$,

$$\left[\sum_{s\in I_2}\frac{m_s}{n_s}\right] \geqslant \min\left\{m_* + \left[\sum_{s\in K}\frac{m_s}{n_s}\right],\ m-1\right\} = m-1.$$

The proof is ended.

Corollary 6. *If $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ covers an $AP_n(|S|)$ at least m times where $S=\{\{\Sigma_{s\in I}m_s/n_s\}:\ I\subseteq\{1,\cdots,k\}\}$ and $m_1,\cdots,m_k$ are positive multiples of $(n,n_1),\cdots,(n,n_k)$ respectively, then for any $J\subseteq\{1,\cdots,k\}$ we have*

$$(28)\qquad \sum_{s\in I}\frac{m_s}{n_s}-\min\left\{\sum_{s\in J}\frac{m_s}{n_s},m-1+\left\{\sum_{s\in J}\frac{m_s}{n_s}\right\}\right\}\in\mathbb{N}$$
$$\text{for some } I\subseteq\{1,\cdots,k\} \text{ with } I\neq J.$$

In particular, when (1) forms an m-cover of $\mathbb{Z}$ (28) holds for any $m_1,\cdots,m_k\in\mathbb{Z}^+$ and $J\subseteq\{1,\cdots,k\}$.

Proof. Immediate from Theorem 2 since $[\Sigma_{s\in L}m_s/n_s]=[\Sigma_{s\in J}m_s/n_s]=m_*+[\Sigma_{s\in K}m_s/n_s]$ if $m_*=0$ and $K=L=J$.

Remark. This corollary improves the extension of Zhang's result mentioned after Lemma 4.

From Corollary 6 we can deduce

Corollary 7. *Let (1) be an m-cover of $\mathbb{Z}$ and $c_1,\cdots,c_k$ be positive integers.*

(i) If $1\leqslant t\leqslant k$, $J\subseteq\{1,\cdots,k\}\setminus\{t\}$ and $(n,n_t)>1$, then

$$(29)\qquad \sum_{s\in I}c_s\frac{(n,n_s)}{n_s}-\min\left\{\sum_{s\in J}c_s\frac{(n,n_s)}{n_s},\ m-1+\left\{\sum_{s\in J}c_s\frac{(n,n_s)}{n_s}\right\}\right\}\in\mathbb{N}$$
$$\text{for some } I\subseteq\{1,\cdots,k\} \text{ with } t\notin I\neq J.$$

(ii) Assume that $\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ n_s\mid n}}^k$ fails to be an m-cover of $\mathbb{Z}$. Then for any $J\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid n\}$ we have

$$(30)\qquad \sum_{s\in I}c_s\frac{(n,n_s)}{n_s}-\sum_{s\in J}c_s\frac{(n,n_s)}{n_s}\in\mathbb{Z} \text{ for some } J\neq I\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid n\},$$

in particular

$$(31)\qquad \sum_{s\in I}c_s\frac{(n,n_s)}{n_s}\in\mathbb{Z}^+\quad \text{for some } I\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid n\}.$$

Proof. i) Let $1\leqslant t\leqslant k$, $t\notin J\subseteq\{1,\cdots,k\}$ and $(n,n_t)>1$. As $1+a_t+n\mathbb{Z}\cap a_t+n_t\mathbb{Z}=\emptyset$, system $\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ s\neq t}}^k$ forms an m-cover of $1+a_t+n\mathbb{Z}$. Applying Corollary 6 we obtain (29).

ii) As in the proof of Corollary 3 there exists an integer a such that system $B=\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ n_s\mid n}}^k$ covers $a+([n_s]_{\substack{s=1\\ n_s\nmid n}}^k,[n_s]_{\substack{s=1\\ n_s\nmid n}}^k)\mathbb{Z}$. Since B is a cover of $a+(n,[n_s]_{\substack{s=1\\ n_s\nmid n}}^k)\mathbb{Z}$, it follows from Corollary 6 that (30) holds for any $J\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid n\}$. When $J=\emptyset$, (30) implies (31). This completes the proof.

Clearly the second part of Corollary 7 yields the latter sentence in part (ii) of Theorem I.

Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be a minimal m-cover of $\mathbb{Z}$ with $N=[n_1,\cdots,n_k]$ not dividing n. Then $B=\{a_s+n_s\mathbb{Z}\}_{\substack{s=1\\ n_s\nmid n}}^k$ doesn't form an m-cover of $\mathbb{Z}$ and so part (ii) of Corollary 6 can be applied. We note that

$$|\{1\leqslant s\leqslant k:\ n_s\nmid n\}|=|\{1\leqslant s\leqslant k:\ n_s\nmid (n,N)\}|\geqslant 1+f(N/(n,N))$$

by the result concerned with (3). Let $p^\alpha \| N$ where p is a prime divisor of N, then for any $\beta = 1, \cdots, \alpha$ we have

$$|\{1 \leqslant s \leqslant k:\ n_s \nmid p^{\beta-1}N/p^\alpha\}| = |\{1 \leqslant s \leqslant k:\ p^\beta \mid n_s\}| \geqslant 1 + (\alpha - \beta + 1)(p-1)$$

and

$$\frac{(p^{\beta-1}N/p^\alpha, n_s)}{n_s} = \frac{1}{p^{\gamma-\beta+1}} \quad \text{if } p^\gamma \| n_s \text{ with } \gamma \geqslant \beta.$$

Let δ be the unique integer for which $p^{\alpha-\delta} \| [n_s]_{\substack{s=1 \\ p^\alpha \nmid n_s}}^k$, then $l = |\{1 \leqslant s \leqslant k:\ p^\alpha \| n_s\}| \geqslant p^\delta$ if and only if for any $c_1, \cdots, c_k \in \mathbb{Z}^+$ there is an $I \subseteq \{1 \leqslant s \leqslant k:\ n_s \nmid N/p^\delta\}$ with $\Sigma_{s \in I} c_s (N/p^\delta, n_s)/n_s = \Sigma_{s \in I} c_s/p^\delta \in \mathbb{Z}^+$. (Observe that if $l \geqslant p^\delta$ and $\{1 \leqslant s \leqslant k:\ p^\alpha \| n_s\} = \{i_1, \cdots, i_l\}$ then the $l+1$ numbers

$$0,\ c_{i_1},\ c_{i_1} + c_{i_2},\ \cdots,\ c_{i_1} + \cdots + c_{i_l}$$

cannot have distinct residues modulo p^δ.)

Corollary 8. *Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ and $c_1, \cdots, c_{t-1}, c_{t+1}, \cdots, c_k \in \mathbb{Z}^+$ where $1 \leqslant t \leqslant k$.*

(i) Suppose that A covers

$$2\left|\left\{\left\{\sum_{s \in I} c_s \frac{(n, n_s)}{n_s}\right\}:\ I \subseteq \{s \in J:\ s \neq t\}\right\}\right|$$

consecutive terms in $a + n\mathbb{Z}$ at least m times where $a \in \mathbb{Z}$ and $J = \{1 \leqslant s \leqslant k:\ (n, n_s) \mid a_s - a\}$. When $m > 1$,

$$\sum_{s \in I} c_s \frac{(n, n_s)}{n_s} \in \mathbb{Z}^+ \quad \textit{for some } I \subseteq J \setminus \{t\}. \tag{32}$$

For $m = 1$, providing $\Sigma_{s \in I} c_s (n, n_s)/n_s \notin \mathbb{Z}^+$ for any $I \subseteq J \setminus \{t\}$ we have

$$\left\{\left\{\sum_{s \in I} c_s \frac{(n, n_s)}{n_s}\right\}:\ I \subseteq J \setminus \{t\}\right\} \supseteq \left\{\frac{(n, n_t)}{n_t} r:\ r = 0, 1, \cdots, \frac{n_t}{(n, n_t)} - 1\right\}. \tag{33}$$

(ii) Assume that A is an m-cover of $\mathbb{Z}$. If $m > 1$ and $(n, n_j) > 1$ with $1 \leqslant j \leqslant k$, then (32) holds for $J = \{1, \cdots, k\} \setminus \{j\}$. When $\{a_s + n_s\mathbb{Z}\}_{\substack{s=1 \\ s \notin J}}^k$ fails to be an m-cover of $\mathbb{Z}$ where $J \subseteq \{1, \cdots, k\}$ and $[n_s]_{s \notin J} \mid n$, either (32) or (33) is true.

Proof. i) Fix $r \in \{0, 1, \cdots, n_t/(n, n_t) - 1\}$ and set $c_t = r + mn_t/(n, n_t)$. Let $m_s = c_s(n, n_s)$ for every $s \in J$. Obviously $A' = \{a_s + n_s\mathbb{Z}\}_{s \in J}$ covers

$$\left|\left\{\left\{\sum_{s \in I} \frac{m_s}{n_s}\right\}:\ I \subseteq J\right\}\right| \quad \left(\leqslant 2\left|\left\{\left\{\sum_{s \in I} \frac{m_s}{n_s}\right\}:\ I \subseteq J \setminus \{t\}\right\}\right|\right)$$

consecutive terms in $a + n\mathbb{Z}$ at least m times, since $a_s + n_s\mathbb{Z} \cap a + n\mathbb{Z} \neq \emptyset$ if and only if $s \in J$. Applying Corollary 6 to A' we obtain that

$$\sum_{s \in I} c_s \frac{(n, n_s)}{n_s} = \sum_{s \in I} \frac{m_s}{n_s} \in \mathbb{N} \quad \text{for some nonempty } I \subseteq J,$$

so (32) holds if $t \notin J$. From now on we assume $t \in J$. By Corollary 6 for some $I(r) \subseteq J$ with $I(r) \neq \{t\}$ we have

$$\sum_{s\in I(r)} \frac{m_s}{n_s} - \min\left\{\frac{m_t}{n_t},\ m-1+\left\{\frac{m_t}{n_t}\right\}\right\} \in \mathbb{N},$$

i.e.

$$(*)\qquad \sum_{s\in I(r)} c_s\frac{(n,n_s)}{n_s} - \left(m-1+\frac{(n,n_t)}{n_t}r\right) \in \mathbb{N}.$$

Let's assume that $\Sigma_{s\in I} c_s(n,n_s)/n_s \notin \mathbb{Z}^+$ for every $I \subseteq J\setminus\{t\}$. Then for each $r = 0, 1, \cdots, n_t/(n,n_t)-1$, $t \notin I(r)$ and therefore $I(r) \subseteq J\setminus\{t\}$ since otherwise we would have

$$\sum_{s\in I'(r)} c_s\frac{(n,n_s)}{n_s} \in \mathbb{Z} \ \text{ where } \emptyset \neq I'(r) = I(r)\setminus\{t\} \subseteq J\setminus\{t\}$$

which contradicts our assumption. As

$$\sum_{s\in I(0)} c_s\frac{(n,n_s)}{n_s} \notin \mathbb{Z}^+ \ \text{ and } \ \sum_{s\in I(0)} c_s\frac{(n,n_s)}{n_s} - \left(m-1+\frac{(n,n_t)}{n_t}\cdot 0\right) \in \mathbb{N},$$

we must have $m = 1$ and $I(0) = \emptyset$. So $(*)$ can be restated as follows:

$$\left\{\sum_{s\in I(r)} \frac{c_s}{n_s/(n,n_s)}\right\} = \frac{r}{n_t/(n,n_t)}.$$

This proves part (i).

ii) If $1 \leqslant j \leqslant k$ and $(n,n_j) > 1$, then $j \notin \{1 \leqslant s \leqslant k :\ (n,n_s) \mid 1+a_j-a_s\}$ and A forms an m-cover of $1+a_j+n\mathbb{Z}$. If $\{a_s+n_s\mathbb{Z}\}_{s=1, s\notin J}^k$ isn't an m-cover of $\mathbb{Z}$ where $J \subseteq \{1,\cdots,k\}$ and $[n_s]_{s\notin J} \mid n$, then as in the proof of Corollary 3 system $B = \{a_s+n_s\mathbb{Z}\}_{s\in J}$ covers $a+(n,[n_s]_{s\in J})\mathbb{Z} \subseteq a+([n_s]_{s\notin J},[n_s]_{s\in J})\mathbb{Z}$ for some integer a. Now part (ii) comes true if we apply part (i) to A and B.

Remark. With an m-cover (1) of $\mathbb{Z}$ given, the first sentence in part (ii) of Theorem I follows from part (i) of Corollary 8 in the case $n = 1$. If $A = \{a_s+n_s\mathbb{Z}\}_{s=1}^k$ covers all the integers then by part (i) of Corollary 8 for any $m_1,\cdots,m_k \in \mathbb{Z}^+$ we have either (11) or

$$(34)\qquad \left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I \subseteq \{1,\cdots,k\}\setminus\{t\}\right\} \supseteq \left\{0, \frac{1}{n_t}, \cdots, \frac{n_t-1}{n_t}\right\},$$

As an important complement to part (ii) of Theorem 1 and Corollary 5 we give here

Corollary 9. *If $\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ is an exact cover of $a+d\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential where $a\in\mathbb{Z}$, $d\mid n$ and $1\leqslant t\leqslant k$, then*

$$\begin{aligned}&\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid d\ \&\ s\neq t\}\right\}\\ \supseteq&\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq\{1\leqslant s\leqslant k:\ (d,n_s)\mid a_s-a\ \&\ s\neq t\}\right\}\\ \supseteq&\left\{\frac{(n,n_t)}{n_t}r:\ r=0,1,\cdots,\frac{n_t}{(n,n_t)}-1\right\}.\end{aligned}$$

Thus, when (1) forms an exact cover of $\mathbb{Z}$, for every $t=1,\cdots,k$ we have

$$\begin{aligned}&\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq\{1\leqslant s\leqslant k:\ n_s\nmid n\}\setminus\{t\}\right\}\\ \supseteq&\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq\{1\leqslant s\leqslant k:\ (n,n_s)\mid a_s-a_t\ \&\ s\neq t\}\right\}\\ \supseteq&\left\{\frac{(n,n_t)}{n_t}r:\ r=0,1,\cdots,\frac{n_t}{(n,n_t)}-1\right\}.\end{aligned}\tag{35}$$

Proof. If (1) is an exact cover of $\mathbb{Z}$ then (1) forms an exact cover of $a_t+n\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential since any $a_s+n_s\mathbb{Z}$ with $s\neq t$ doesn't contain a_t. So it suffices to confirm the first sentence in Corollary 9.

Let $a\in\mathbb{Z},\ d\mid n$ and assume that $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ is an exact cover of $a+d\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential. Since $a+d\mathbb{Z}$ can be partitioned into $a+rd+n\mathbb{Z},\ r=0,1,\cdots,n/d-1$, for some $b\equiv a\pmod d$ system A forms an exact cover of $b+n\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential. Clearly $J=\{1\leqslant s\leqslant k:\ (n,n_s)\mid a_s-b\}$ contains t, and $\{q_j+(n_j/(n,n_j))\mathbb{Z}\}_{j\in J}$ partitions $\mathbb{Z}$ by Lemma 1 where each q_j is an integer with $b+nq_j\equiv a_j\pmod{n_j}$. Thus

$$\sum_{j\in J\setminus\{t\}}\frac{(n,n_j)}{n_j}<\sum_{j\in J}\frac{1}{n_j/(n,n_j)}=1$$

and therefore

$$\sum_{s\in I}\frac{(n,n_s)}{n_s}\notin\mathbb{Z}^+\quad\text{for any } I\subseteq J\setminus\{t\}.$$

In view of part (i) of Corollary 8,

$$\begin{aligned}&\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq J\setminus\{t\}\right\}=\left\{\left\{\sum_{s\in I}\frac{(n,n_s)}{n_s}\right\}:\ I\subseteq J\setminus\{t\}\right\}\\ \supseteq&\left\{\frac{(n,n_t)}{n_t}r:\ r=0,1,\cdots,\frac{n_t}{(n,n_t)}-1\right\}.\end{aligned}$$

Notice that $J\subseteq\{1\leqslant s\leqslant k:\ (d,n_s)\mid a_s-a\}$. We also have

$$\{1\leqslant s\leqslant k:\ (d,n_s)\mid a_s-a\ \&\ s\neq t\}\cap\{1\leqslant s\leqslant k:\ n_s\mid d\}=\emptyset,$$

for, if $(d,n_s)\mid a_s-a$ and $n_s\mid d$ then $a+d\mathbb{Z}\subseteq a_s+n_s\mathbb{Z}$ and so $s=t$ since $a+d\mathbb{Z}\cap a_t+n_t\mathbb{Z}\neq\emptyset$ and A covers $a+d\mathbb{Z}$ exactly once. This concludes the proof.

Observe that $[n_1,\cdots,n_k]^{-1}\mathbb{Z}=n_1^{-1}\mathbb{Z}+\cdots+n_k^{-1}\mathbb{Z}$. In contrast with the false conjecture of Z. H. Sun mentioned in Section 3, putting $n=1$ in Corollary 9 we obtain part (iii) of Theorem II, which seems beautiful and important. Note that in the first example $1/8+5/12=13/24\notin S(A_1)$ though $1/8,5/12\in S(A_1)$.

Now let's say something similar to Corollary 9 for general (possibly not exact) m-covers of an arithmetic sequence.

Corollary 10. *Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ and $t\in\{1,\cdots,k\}$. For each $s=1,\cdots,k$ with $s\neq t$ we let m_s be a positive integer with $(m_s,n_s)=(n,n_s)$.*

(i) If for some integer a and divisor d of n system A forms an m-cover of $a+d\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential, then

$$\left\{\left\{\sum_{s\in I_1}\frac{m_s}{n_s}-\sum_{s\in I_2}\frac{m_s}{n_s}\right\}:\ I_1,I_2\subseteq J\setminus\{t\},\ \sum_{s\in I_1}\frac{m_s}{n_s}\geqslant m-1\ \&\ \sum_{s\in I_2}\frac{m_s}{n_s}\geqslant m-2\right\}$$

contains

$$\left\{\frac{(n,n_t)}{n_t}r:\ r=0,1,\cdots,\frac{n_t}{(n,n_t)}-1\right\}$$

where $J=\{1\leqslant s\leqslant k:\ (d,n_s)\mid a_s-a\}$.

(ii) Assume that A is an m-cover of $\mathbb{Z}$ but $A_t=\{a_s+n_s\mathbb{Z}\}_{s=1,\, s\neq t}^k$ is not. Then for every integer r there exist $I_1,I_2\subseteq\{1,\cdots,k\}\setminus\{t\}$ for which

$$|I_i\cap\{1\leqslant s\leqslant k:\ n_s\mid n\}|<m\ \&\ \sum_{s\in I_i}\frac{m_s}{n_s}\geqslant m-i\ \text{for}\ i=1,2 \tag{36}$$

and

$$\frac{(n,n_t)}{n_t}r\equiv\sum_{s\in I_1}\frac{m_s}{n_s}-\sum_{s\in I_2}\frac{m_s}{n_s}\pmod 1. \tag{37}$$

Proof. i) Fix an integer r with $0<r\leqslant n_t/(n,n_t)$. Set $m_t=r(n,n_t)$, $m_*=m-1$, $K=\emptyset$ and $L=J\setminus\{t\}$. Then simply apply Theorem 2 to $A'=\{a_s+n_s\mathbb{Z}\}_{s\in J}$.

ii) For some $a=0,1,\cdots,n-1$ system A forms an m-cover of $a+n\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential. By part (i) it suffices to show that

$$T=\{1\leqslant s\leqslant k:\ (n,n_s)\mid a_s-a\ \&\ s\neq t\}\cap\{1\leqslant s\leqslant k:\ n_s\mid n\}$$

has cardinality less than m. In fact, $a+n\mathbb{Z}\subseteq a_s+n_s\mathbb{Z}$ for each $s\in T$, if $|T|\geqslant m$ then A_t would also be an m-cover of $a+n\mathbb{Z}$, a contradiction! We are done.

Remark. Part (iii) of Theorem I follows from the second parts of Corollary 10 and Theorem 1. If (1) is an exact m-cover of $\mathbb{Z}$ then for any $t=1,\cdots,k$ and $r=0,1,\cdots,n_t-1$ by part (ii) of Corollary 10

$$\frac{r}{n_t}\equiv\sum_{s\in I_1}\frac{1}{n_s}+\sum_{s\in I_2}\frac{1}{n_s}\ (\text{mod } 1)\ \text{for some}\ I_1,I_2\subseteq\{1,\cdots,k\}\setminus\{t\} \tag{38}$$

since for $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ and $I'=(\{1,\cdots,k\}\setminus\{t\})\setminus I$ we obviously have

$$-\sum_{s\in I}\frac{1}{n_s}=-\sum_{\substack{s=1\\ s\neq t}}^k\frac{1}{n_s}+\sum_{s\in I'}\frac{1}{n_s}=\frac{1}{n_t}-m+\sum_{s\in I'}\frac{1}{n_s}.$$

In the case $m=1$ we can fix $I_2=\emptyset$ (by part (iii) of Theorem II).

Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be a cover of $\mathbb{Z}$ with $a_t + n_t\mathbb{Z}$ essential. Clearly A covers some $AS(n)$ with $a_t + n_t\mathbb{Z}$ essential. Let

$$S_{n,t} = \left\{\sum_{s\in I}\frac{c_s}{n_s/(n,n_s)} :\ I \subseteq \{1 \leqslant s \leqslant k :\ s \neq t \ \&\ n_s \nmid n\}\right\} \tag{39}$$

where each c_s is a positive integer prime to $n_s/(n, n_s)$. Part (ii) of Theorem 1 gives that $|\{\{x\} :\ x \in S_{n,t}\}| \geqslant n_t/(n, n_t)$, and the second part of Corollary 10 tells that

$$\{\{x - y\} :\ x, y \in S_{n,t}\} \supseteq \left\{\frac{r}{n_t/(n,n_t)} :\ r = 0, 1, \cdots, \frac{n_t}{(n,n_t)} - 1\right\} \tag{40}$$

which is connected with difference sets.

Corollaries 9 and 10 lead us to calculate the cardinality of

$$D_n(A) = \bigcup_{t=1}^{k}\left\{\frac{(n,n_t)}{n_t}r :\ r = 0, 1, \cdots, \frac{n_t}{(n,n_t)} - 1\right\}$$

which sometimes serves as a lower bound. Let $N = [n_1, \cdots, n_k]$. Obviously

$$D_n(A) = \bigcup_{\substack{d \mid n_s/(n,n_s)\\ \text{for some } s=1,\cdots,k}} \left\{\frac{c}{d} :\ c \in \mathbb{Z},\ 0 \leqslant c < d \text{ and } (c, d) = 1\right\}$$

consists of those $x(n, N)/N$ with $x \in \mathbb{Z}$ and $x(n, N)/N = r(n, n_s)/n_s$ for some $s = 1, \cdots, k$ and $r = 0, 1, \cdots, n_s/(n, n_s) - 1$. Thus

$$\begin{aligned} D_n(A) &= \left\{\frac{x}{N/(n,N)} :\ 0 \leqslant x < \frac{N}{(n,N)},\ x \in \frac{N/(n,N)}{n_s/(n,n_s)}\mathbb{Z} \text{ for some } s = 1, \cdots, k\right\} \\ &= \left\{\frac{x}{[n,N]/n} :\ 0 \leqslant x < \frac{[n,N]}{n} \text{ and } x \in \bigcup_{s=1}^{k}\frac{[n,N]}{[n,n_s]}\mathbb{Z}\right\} \end{aligned}$$

and so

$$\begin{aligned} |D_n(A)| &= \sum_{\substack{d \mid n_s/(n,n_s)\\ \text{for some } s=1,\cdots,k}} \varphi(d) \\ &= \left|\left\{0 \leqslant x < \frac{[n,N]}{n} :\ x \in \bigcup_{s=1}^{k}\frac{[n,N]}{[n,n_s]}\mathbb{Z}\right\}\right| = \frac{[n,N]}{n} d\left(\bigcup_{s=1}^{k}\frac{[n,N]}{[n,n_s]}\mathbb{Z}\right) \end{aligned}$$

where as in (5) $d(\cdot)$ represents the asympotic density of a set.

5. Number of $I \subseteq \{1, \cdots, k\}$ such that $\sum_{s\in I}\frac{m_s}{n_s} = v$

Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$. Let $m_1, \cdots, m_k \in \mathbb{Z}^+$ and $v \in \{\Sigma_{s\in I} m_s/n_s :\ I \subseteq \{1, \cdots, k\}\}$. By Corollary 6,

$$\left|\left\{I \subseteq \{1, \cdots, k\} :\ \sum_{s\in I}\frac{m_s}{n_s} - v \in \mathbb{Z}\right\}\right| > 1.$$

When $v = 0$ or $\Sigma_{s=1}^k m_s/n_s$, this gives that $\Sigma_{s\in I} m_s/n_s \in \mathbb{Z}^+$ for some $I \subseteq \{1, \cdots, k\}$. When $m_1 = \cdots = m_k = 1$ and A forms an exact m-cover of $\mathbb{Z}$, if

$0 < v < \Sigma_{s=1}^{k} m_s/n_s = m$ then by Theorem 4 of [21]

$$\left|\left\{I \subseteq \{1, \cdots, k\} :\ \sum_{s \in I} \frac{m_s}{n_s} = v\right\}\right| > 1,$$

and moreover by the main result of [20]

$$\left|\left\{I \subseteq \{1, \cdots, k\} :\ \sum_{s \in I} \frac{m_s}{n_s} = v\right\}\right| \geqslant \binom{m}{v} \qquad \text{if } v \in \{0, 1, \cdots, m\}.$$

In this section we'll say something further on $|\{I \subseteq \{1, \cdots, k\} :\ \Sigma_{s \in I} m_s/n_s = v\}|$.

Lemma 6. *For any numbers* $a_1, \cdots, a_n$ *and* $x_1, \cdots, x_n$,

$$\sum_{t=1}^{n} \binom{a_t}{l} x_t = 0 \quad \text{for every } l = 0, 1, \cdots, m-1$$

if and only if

$$\sum_{t=1}^{n} a_t^l x_t = 0 \quad \text{for all } l = 0, 1, \cdots, m-1.$$

Proof. For each $l = 0, 1, \cdots, m-1$,

$$\binom{x}{l} = \sum_{j=0}^{l} (-1)^{l-j} \frac{s(l,j)}{l!} x^j \quad \text{and} \quad x^l = \sum_{j=0}^{l} j! S(l,j) \binom{x}{j}$$

where $s(l,j)$ and $S(l,j)$ $(0 \leqslant j \leqslant l)$ are Stirling numbers of the first kind and the second kind respectively. From these we can easily deduce Lemma 6.

Lemma 7. *Let* $a_1, \cdots, a_{\max\{m,n\}}$ *be distinct numbers. Then*

$$\sum_{t=1}^{n} a_t^{s-1} x_t = 0 \quad \text{for every } s = 1, \cdots, m$$

if and only if

$$\sum_{t=1}^{n} a_{st} x_t = 0 \quad \text{for all } s = 1, \cdots, m$$

where

$$a_{st} = \prod_{\substack{i=1 \\ i \neq s}}^{m} \frac{a_i - a_t}{a_i - a_s} \quad \text{for } s = 1, \cdots, m \text{ and } t = 1, \cdots, n.$$

Proof. When $n \geqslant m$, this is just Lemma 4 of [21]. In the case $n < m$,

$$\begin{aligned}
&\sum_{t=1}^{n} a_t^{s-1} x_t = 0 \ \text{ for every } s = 1, \cdots, m \\
\Longrightarrow &\sum_{t=1}^{n} a_t^{s-1} x_t = 0 \ \text{ for every } s = 1, \cdots, n \\
\Longrightarrow &x_1 = \cdots = x_n = 0 \ \text{ (Vandermonde)} \\
\Longrightarrow &\sum_{t=1}^{n} a_t^{s-1} x_t = 0 \ \text{ for every } s = 1, 2, 3, \cdots
\end{aligned}$$

and

$$\begin{aligned}&\sum_{t=1}^{n} a_{st}x_t = 0 \ \text{ for all } s = 1, \cdots, m\\ \Longleftrightarrow &\sum_{t=1}^{n} \delta_{st}x_t = 0 \ \text{ for all } s = 1, \cdots, m\\ \Longleftrightarrow & x_t = 0 \ \text{ for all } t = 1, \cdots, n\end{aligned}$$

where δ_{st} is the Kronecker delta. Thus the desired result follows.

Remark. The original ideas for Lemmas 6 and 7 can be found in [21].

As a consequence of Lemmas 4, 6 and 7 we have

Lemma 8. *Let $a \in \mathbb{Z}^+$, $J = \{1 \leqslant s \leqslant k : \ (n, n_s) \mid a_s - a\}$ and $m_j \in \mathbb{Z}^+$ for all $j \in J$. Let S be as in (26) and q_j, $j \in J$ be as in Lemma 5. For each $\theta \in S$ we let $U(\theta)$ be a set of m distinct numbers comparable with*

$$V(\theta) = \left\{ \sum_{s\in I} \frac{m_s}{n_s} : \ I \subseteq J \ \textit{and} \ \left\{ \sum_{s\in I} \frac{m_s}{n_s} \right\} = \theta \right\}$$

(i.e. $|U(\theta)| = m$, and either $U(\theta) \subseteq V(\theta)$ or $U(\theta) \supseteq V(\theta)$), and $T(\theta)$ a nonempty set of reals in $[0, 1)$ comparable with

$$W(\theta) = \left\{ \left\{ \sum_{s\in I} \frac{m_s}{n_s} q_s \right\} : \ I \subseteq J \ \textit{and} \ \left\{ \sum_{s\in I} \frac{m_s}{n_s} \right\} = \theta \right\}$$

for which $|T(\theta)| < d(\sigma - \theta)$ for all $\sigma \in S \setminus \{\theta\}$ where we use $d(r)$ to denote the denominator d of a rational $r = c/d$ with $c \in \mathbb{Z}$, $d \in \mathbb{Z}^+$ and $(c, d) = 1$. Then part (ii) of Lemma 5 holds if and only if for any $\theta \in S$ one has

$$\sum_{\substack{v\in V(\theta)\\ w\in W(\theta)}} a_{uv}b_{vw}c_{wt} = 0 \quad \textit{for all } u \in U(\theta) \textit{ and } t \in T(\theta) \tag{41}$$

where

$$a_{uv} = \prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x - v}{x - u}, \quad b_{vw} = \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v\\ \{\Sigma_{s\in I} q_s m_s/n_s\} = w}} (-1)^{|I|}$$

and

$$c_{wt} = e^{2\pi i w} \prod_{\substack{x\in T(\theta)\\ x\neq t}} \frac{e^{2\pi i x} - e^{2\pi i w}}{e^{2\pi i x} - e^{2\pi i t}}.$$

Proof. Let $\theta\in S$ and $l\in\{1,\cdots,|T(\theta)|\}$. For each $j=0,1,\cdots,m-1$, by the fact that $d(\sigma-\theta)\nmid l$ for $\sigma\in S\setminus\{\theta\}$, we have

$$\begin{aligned}&\sum_{\substack{I\subseteq J\\ l(\Sigma_{s\in I}m_s/n_s-\theta)\in\mathbb{Z}}}(-1)^{|I|}\left(l\sum_{s\in I}\frac{m_s}{n_s}\right)^j e^{2\pi i l\Sigma_{s\in I}q_sm_s/n_s}=0\\ \iff&\sum_{v\in V(\theta)}v^j\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v}}(-1)^{|I|}e^{2\pi i l\Sigma_{s\in I}q_sm_s/n_s}\\ &=\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}}(-1)^{|I|}\left(\sum_{s\in I}\frac{m_s}{n_s}\right)^j e^{2\pi i l\Sigma_{s\in I}q_sm_s/n_s}=0.\end{aligned}$$

As

$$\binom{\Sigma_{s\in I}lm_s/n_s}{j}=\sum_{i=0}^{j}\binom{[\Sigma_{s\in I}lm_s/n_s]}{i}\binom{\{\Sigma_{s\in I}lm_s/n_s\}}{j-i}$$

and

$$\binom{[\Sigma_{s\in I}lm_s/n_s]}{j}=\sum_{i=0}^{j}\binom{\Sigma_{s\in I}lm_s/n_s}{i}\binom{-\{\Sigma_{s\in I}lm_s/n_s\}}{j-i}$$

for all $I\subseteq J$, by Lemmas 6, 7 and the above

$$(\star)\qquad\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}lm_s/n_s\}=\{l\theta\}}}(-1)^{|I|}\binom{[\Sigma_{s\in I}\frac{lm_s}{n_s}]}{j}e^{\Sigma_{s\in I}q_slm_s/n_s}=0\quad\text{for } j=0,\cdots,m-1$$

holds if and only if

$$\sum_{\substack{v\in V(\theta)\\ w\in W(\theta)}}a_{uv}b_{vw}e^{2\pi i lw}=\sum_{v\in V(\theta)}\left(\prod_{\substack{x\in U(\theta)\\ x\neq u}}\frac{x-v}{x-u}\right)\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v}}(-1)^{|I|}e^{2\pi i l\Sigma_{s\in I}q_sm_s/n_s}$$

vanishes for every $u\in U(\theta)$.

Under part (ii) of Lemma 5, system $\{q_s+(n_s/m_s)\mathbb{Z}\}_{s\in J}$ forms an m-cover of $\mathbb{Z}$ by Lemma 4 and hence $\{q_s+(lm_s/n_s)^{-1}\mathbb{Z}\}_{s\in J}$ forms an m-cover of $\mathbb{Z}$ for every positive integer l, therefore by Lemma 4 ($\star$) is valid for any $\theta\in S$ and $l=1,\cdots,|T(\theta)|,|T(\theta)|+1,\cdots$. On the other hand, that ($\star$) holds for $l=1$

and $\theta \in S$ gives part (ii) of Lemma 5. So, by the above and Lemma 7,

$$\begin{aligned} &\text{part (ii) of Lemma 5 holds}\\ \iff &(\star) \text{ is true for each } \theta \in S \text{ and } l = 1, \cdots, |T(\theta)|\\ \iff &\sum_{w \in W(\theta)} (e^{2\pi i w})^{l-1} \sum_{v \in V(\theta)} a_{uv} b_{vw} e^{2\pi i w} = \sum_{\substack{v \in V(\theta)\\ w \in W(\theta)}} a_{uv} b_{vw} e^{2\pi i l w} = 0\\ &\quad \text{for any } \theta \in S,\ u \in U(\theta) \text{ and } l = 1, \cdots, |T(\theta)|\\ \iff &\sum_{w \in W(\theta)} \Bigg(\prod_{\substack{x \in T(\theta)\\ x \neq t}} \frac{e^{2\pi i x} - e^{2\pi i w}}{e^{2\pi i x} - e^{2\pi i t}} \Bigg) \sum_{v \in V(\theta)} a_{uv} b_{vw} e^{2\pi i w} = 0\\ &\quad \text{for all } \theta \in S,\ u \in U(\theta) \text{ and } t \in T(\theta)\\ \iff &(41) \text{ holds for every } \theta \in S. \end{aligned}$$

This ends our proof.

One more technical lemma is needed.

Lemma 9. *Let $c_1, \cdots, c_k \in \mathbb{N}$ and $d_1, \cdots, d_l \in \mathbb{Z}^+$. Suppose that there exist nonzero numbers $x_1, \cdots, x_k$ such that $\Sigma_{s=1}^k c_s x_s^t = 0$ for those $t \in \mathbb{Z}^+$ divisible by none of $d_1, \cdots, d_l$. Then*

$$c_1 + \cdots + c_k \in d_l\mathbb{N} + \cdots + d_l\mathbb{N},$$

in particular $(d_1, \cdots, d_l) \mid c_1 + \cdots + c_k$.

Proof. Since each c_s is the sum of finitely many 1's, without loss of generality we may assume that $c_1 = \cdots = c_k = 1$. Let

$$D = d_1\mathbb{N} + \cdots + d_l\mathbb{N} = \{d_1 z_1 + \cdots + d_l z_l :\ z_1, \cdots, z_l \in \mathbb{N}\}$$

and

$$P(x) = \prod_{s=1}^k (x - x_s) = \sum_{i=0}^k (-1)^i \sigma_i x^{k-i}.$$

We claim that for every $j = 0, 1, \cdots, k$ either $j \in D$ or $\sigma_j = 0$.

Clearly $0 \in D$. Let $1 \leqslant j \leqslant k$ and assume that for any $i = 0, 1, \cdots, j-1$ either $i \in D$ or $\sigma_i = 0$. By Newton's symmetric functions identity (cf. [9])

$$\sum_{i=0}^{j-1} (-1)^i \sigma_i (x_1^{j-i} + \cdots + x_k^{j-i}) + (-1)^j \sigma_j j = 0$$

and hence

$$\begin{aligned} (-1)^j j \sigma_j &= -\sum_{i=0}^{j-1} (-1)^i \sigma_i (x_1^{j-i} + \cdots + x_k^{j-i})\\ &= -\sum_{\substack{i=0\\ i \in D}}^{j-1} (-1)^i \sigma_i (x_1^{j-i} + \cdots + x_k^{j-i}) \ \ (\text{since } \sigma_i = 0 \text{ if } i \notin D)\\ &= -\sum_{\substack{i=0\\ i \in D,\ j-i \in D}}^{j-1} (-1)^i \sigma_i (x_1^{j-i} + \cdots + x_k^{j-i}) \ \left(\sum_{s=1}^k x_s^{j-i} = 0 \text{ if } j - i \notin D \right). \end{aligned}$$

So if $j \notin D$ then we must have $\sigma_j = 0$. This proves the claim by induction.

As $(-1)^k\sigma_k = P(0) \neq 0$, it follows from the claim that

$$k = \sum_{s=1}^{k} 1 \in D \subseteq d_1\mathbb{Z} + \cdots + d_l\mathbb{Z} = (d_1, \cdots, d_l)\mathbb{Z}.$$

The proof is now complete.

Remark. A slightly weaker version of Lemma 9 was proved by Y. G. Chen and Porubský [3].

Now we are able to establish

Theorem 3. *Assume that (1) covers $|S|$ consecutive terms in $a + n\mathbb{Z}$ at least m times where a is an integer, S is the set of those $\{\Sigma_{s\in I} m_s/n_s\}$ with*

$$I \subseteq J = \{1 \leqslant s \leqslant k : \ (n, n_s) \mid a_s - a\} \subseteq \{1, \cdots, k\}$$

and $m_1, \cdots, m_k$ are positive integers divisible by $(n, n_1), \cdots, (n, n_k)$ respectively.

(I) *Let $\theta \in S$ and $P(\theta) = \{I \subseteq J : \ \{\Sigma_{s\in I} m_s/n_s\} = \theta\}$. Set*

$$V(\theta) = \left\{\sum_{s\in I} \frac{m_s}{n_s} : \ I \in P(\theta)\right\} \text{ and } W(\theta) = \left\{\left\{\sum_{s\in I} \frac{m_s}{n_s} q_s\right\} : \ I \in P(\theta)\right\} \tag{42}$$

where for each $s \in J$ we define q_s to be such an integer that $a_s + nq_s \equiv a_s \pmod{n_s}$.

Ia. *For $v \in V(\theta)$ and $w \in W(\theta)$ with $p_0(v, w) \neq p_1(v, w)$ where*

$$p_0(v, w) = \left|\left\{I \subseteq J : \ 2 \mid |I|, \ \sum_{s\in I} \frac{m_s}{n_s} = v \ \& \ \left\{\sum_{s\in I} \frac{m_s}{n_s} q_s\right\} = w\right\}\right|$$

and

$$p_1(v, w) = \left|\left\{I \subseteq J : \ 2 \nmid |I|, \ \sum_{s\in I} \frac{m_s}{n_s} = v \ \& \ \left\{\sum_{s\in I} \frac{m_s}{n_s} q_s\right\} = w\right\}\right|,$$

at least one of the following (i)–(iii) holds.

(i) $|V_w(\theta)| > m$ where

$$V_w(\theta) = \left\{\sum_{s\in I} \frac{m_s}{n_s} : \ I \in P(\theta) \text{ and } \left\{\sum_{s\in I} \frac{m_s}{n_s} q_s\right\} = w\right\}; \tag{43}$$

(ii) $|W_v(\theta)| \geqslant d(\sigma - \theta)$ for some $\sigma \in S \setminus \{\theta\}$ where

$$W_v(\theta) = \left\{\left\{\sum_{s\in I} \frac{m_s}{n_s} q_s\right\} : \ I \in P(\theta) \text{ and } \sum_{s\in I} \frac{m_s}{n_s} = v\right\}; \tag{44}$$

(iii) $|V(\theta)| > m$, and $|W(\theta)| \geqslant d(\sigma - \theta)$ for some $\sigma \in S$ with $\sigma \neq \theta$.

Ib. *If for some $v \in V(\theta)$ all the numbers $p_0(v, w) - p_1(v, w), \ w \in W_v(\theta)$ have the same sign and*

$$\left|\sum_{\substack{I\subseteq J \\ \Sigma_{s\in I} m_s/n_s = v}} (-1)^{|I|}\right| \notin S(\theta) = \left\{\sum_{\substack{\sigma\in S \\ \sigma\neq\theta}} d(\sigma - \theta)x_\sigma : \ x_\sigma \in \mathbb{N} \text{ for } \sigma \in S \setminus \{\theta\}\right\},$$

we then have $|P(\theta)| \geqslant |V(\theta)| > m$. *Providing*

$$\sum_{\substack{I\subseteq J \\ \{\Sigma_{s\in I} m_s/n_s\}=\theta}} (-1)^{|I|} \left[\sum_{s\in I}\frac{m_s}{n_s}\right]^l \neq 0 \quad \text{for some } l=0,1,\cdots,m-1, \tag{45}$$

$|P(\theta)| \geqslant |W(\theta)| \geqslant d(\sigma-\theta)$ *for some* $\sigma \in S \setminus \{\theta\}$.

(II) *For any number* v *we let*

$$\begin{aligned} K(v) &= \left\{ I \subseteq \{1,\cdots,k\} : \sum_{s\in I}\frac{m_s}{n_s} - v \in \mathbb{Z} \right\}, \\ L(v) &= \left\{ I \subseteq \{1,\cdots,k\} : \sum_{s\in I}\frac{m_s}{n_s} = v \right\}. \end{aligned} \tag{46}$$

IIa. *When* $|\{1 \leqslant s \leqslant k : \ n_s \mid m_s\}| < m$, *for some* $v \in \mathbb{Q}$ *we have*

$$|K(v)| \geqslant d\left(\sum_{s\in I}\frac{m_s}{n_s} - v\right) \quad \text{for some } I\subseteq\{1,\cdots,k\} \text{ with } \sum_{s\in I}\frac{m_s}{n_s} - v \notin \mathbb{Z}. \tag{47}$$

IIb. *Let* v *be such a number that*

$$\left| \left\{ \sum_{s\in I}\frac{m_s}{n_s} : \ I \in K(v) \right\} \right| \leqslant m,$$

or every element of $K(v)$ *contains* t *for which (1) forms an exact* m*-cover of* $a+n\mathbb{Z}$ *with* $a_t+n_t\mathbb{Z}$ *essential and* $m_s=(n,n_s)$ *for every* $s=1,\cdots,k$. *If* $2 \nmid |L(v)|$ *then*

$$|L(v)| \geqslant d\left(\sum_{s\in I}\frac{m_s}{n_s} - v\right) \quad \text{for some } I\subseteq\{1,\cdots,k\} \text{ with } \sum_{s\in I}\frac{m_s}{n_s} - v \notin \mathbb{Z}. \tag{48}$$

When all the $|I|$ *with* $I \in L(v)$ *are odd, or all of them are even,*

$$\begin{aligned} &|L(v)| = d_1+\cdots+d_l \text{ for some (not necessarily distinct) } d_1,\cdots,d_l \ (l\geqslant 0) \\ &\quad \text{in the form } d\Big(\sum_{s\in I}\frac{m_s}{n_s} - v\Big) \text{ where } I\subseteq\{1,\cdots,k\} \text{ and } \sum_{s\in I}\frac{m_s}{n_s} - v \notin \mathbb{Z}. \end{aligned} \tag{49}$$

Proof. Let's first prove part (I).

Ia) Clearly $v \in V_w(\theta)$ and $w \in W_v(\theta)$ since

$$b_{vw} = \sum_{\substack{I\subseteq J \\ \Sigma_{s\in I} m_s/n_s = v \\ \{\Sigma_{s\in I} q_s m_s/n_s\}=w}} (-1)^{|I|} = p_0(v,w) - p_1(v,w) \neq 0.$$

When $|V_w(\theta)| \leqslant m$, we can choose a set $U(\theta)$ of m distinct numbers such that either $V_w(\theta) \subseteq U(\theta) \subseteq V(\theta)$ or $V_w(\theta) \subseteq V(\theta) \subseteq U(\theta)$; similarly when $|W_v(\theta)| < d(\sigma-\theta)$ for all $\sigma \in S$ with $\sigma \neq \theta$, we may choose a set $T(\theta) \subseteq [0,1)$ with $|T(\theta)| = \min_{\sigma\in S\setminus\{\theta\}} d(\sigma-\theta)-1$ if $S \neq \{\theta\}$, such that either $W_v(\theta) \subseteq T(\theta) \subseteq W(\theta)$ or $W_v(\theta) \subseteq W(\theta) \subseteq T(\theta)$.

If $|V(\theta)| \leqslant m$ and $|W_v(\theta)| < d(\sigma-\theta)$ for all $\sigma \in S\setminus\{\theta\}$, then $v \in V(\theta) \subseteq U(\theta)$, $w \in W_v(\theta) \subseteq T(\theta)$ (where $U(\theta)$ and $T(\theta)$ are chosen as in the above) and hence by

Lemmas 5 and 8

$$\begin{aligned}0 &= \sum_{v'\in V(\theta)}\Big(\prod_{\substack{x\in U(\theta)\\ x\neq v}}\frac{x-v'}{x-v}\Big)\sum_{w'\in W(\theta)}\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v'\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w'}}(-1)^{|I|}e^{2\pi iw'}\prod_{\substack{x\in T(\theta)\\ x\neq w}}F_{w,w'}(x)\\ &= \sum_{w'\in W(\theta)}\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w'}}(-1)^{|I|}e^{2\pi iw'}\prod_{\substack{x\in T(\theta)\\ x\neq w}}F_{w,w'}(x)\\ &= \sum_{w'\in W_v(\theta)}\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w'}}(-1)^{|I|}e^{2\pi iw'}\prod_{\substack{x\in T(\theta)\\ x\neq w}}F_{w,w'}(x)=e^{2\pi iw}b_{vw}\neq 0\end{aligned}$$

where $F_{w,w'}(x)$ stands for $(e^{2\pi ix}-e^{2\pi iw'})/(e^{2\pi ix}-e^{2\pi iw})$. This contradiction shows that either $|V(\theta)|>m$ or $|W_v(\theta)|\geqslant d(\sigma-\theta)$ for some $\sigma\in S$ with $\sigma\neq\theta$. Similarly, if $|V_w(\theta)|\leqslant m$ and $|W(\theta)|<d(\sigma-\theta)$ for all $\sigma\in S\setminus\{\theta\}$, then $v\in V_w(\theta)\subseteq U(\theta)$, $w\in W(\theta)\subseteq T(\theta)$ and by Lemmas 5 and 8

$$\sum_{w'\in W(\theta)}e^{2\pi iw'}\Big(\prod_{\substack{x\in T(\theta)\\ x\neq w}}F_{w,w'}(x)\Big)\sum_{v'\in V(\theta)}\Big(\prod_{\substack{x\in U(\theta)\\ x\neq v}}\frac{x-v'}{x-v}\Big)\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v'\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w'}}(-1)^{|I|}$$

is zero and equal to

$$e^{2\pi iw}\sum_{v'\in V_w(\theta)}\Big(\prod_{\substack{x\in U(\theta)\\ x\neq v}}\frac{x-v'}{x-v}\Big)\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v'\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w}}(-1)^{|I|}=e^{2\pi iw}b_{vw}$$

which is nonzero, also a contradiction. Thus either $|V_w(\theta)|>m$ or $|W(\theta)|\geqslant d(\sigma-\theta)$ for some $\sigma\in S$ with $\sigma\neq\theta$. This together with the above proves Ia.

Ib) Assume that $|V(\theta)|\leqslant m$ and that $V(\theta)$ has an element v for which either $p_0(v,w)-p_1(v,w)\geqslant 0$ for all $w\in W_v(\theta)$ or $p_0(v,w)-p_1(v,w)\leqslant 0$ for all $w\in W_v(\theta)$. Choose $U(\theta)$ to be a set of m distinct numbers which contains $V(\theta)$. For those $l\in\mathbb{Z}^+$ not divisible by any $d(\sigma-\theta)$ with $\sigma\in S\setminus\{\theta\}$, when $I\subseteq J$ the number $l(\Sigma_{s\in I}m_s/n_s-\theta)$ lies in $\mathbb{Z}$ if and only if $\{\Sigma_{s\in I}m_s/n_s\}=\theta$, so by Lemma 5 and the proof of Lemma 8 we have

$$\begin{aligned}0 &= \sum_{\substack{v'\in V(\theta)\\ w\in W(\theta)}}\Big(\prod_{\substack{x\in U(\theta)\\ x\neq v}}\frac{x-v'}{x-v}\Big)\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v'\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w}}(-1)^{|I|}e^{2\pi ilw}\\ &= \sum_{w\in W(\theta)}\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v\\ \{\Sigma_{s\in I}q_sm_s/n_s\}=w}}(-1)^{|I|}e^{2\pi ilw}=\sum_{w\in W_v(\theta)}b_{vw}e^{2\pi ilw}\end{aligned}$$

and hence

$$(\dagger)\qquad \sum_{w\in W_v(\theta)}|b_{vw}|e^{2\pi ilw}=0$$

since all the $b_{vw} = p_0(v,w) - p_1(v,w),\ w \in W_v(\theta)$ have the same sign. When $S = \{\theta\}$, as (†) holds for each $l = 1, 2, \cdots, |W_v(\theta)|$ by Cramer's rule $|b_{vw}| = 0$ for all $w \in W_v(\theta)$ and therefore

$$\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v}} (-1)^{|I|} = \sum_{w\in W_v(\theta)} b_{vw} = 0 \in S(\theta) = \{0\}.$$

If $S \neq \{\theta\}$, then by Lemma 9

$$\left|\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v}} (-1)^{|I|}\right| = \left|\sum_{w\in W_v(\theta)} b_{vw}\right| = \sum_{w\in W_v(\theta)} |b_{vw}| \in S(\theta).$$

This completes the proof of the first assertion in Ib.

Now suppose that $|W(\theta)| < d(\sigma - \theta)$ for all $\sigma \in S \setminus \{\theta\}$. Clearly we can choose a set $T(\theta)$ for which $W(\theta) \subseteq T(\theta) \subseteq [0, 1)$, and $|T(\theta)| = \min_{\sigma\in S\setminus\{\theta\}} d(\sigma - \theta) - 1$ if $S \neq \{\theta\}$. Let $U(\theta)$ be a set of m numbers comparable with $V(\theta)$. For each $u \in U(\theta)$, by Lemmas 5 and 8 for all $w \in W(\theta)$,

$$\sum_{w'\in W(\theta)} e^{2\pi i w'} \Bigg(\prod_{\substack{x\in T(\theta)\\ x\neq w}} F_{w,w'}(x)\Bigg) \sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x-v}{x-u}\Bigg) \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v\\ \{\Sigma_{s\in I} q_s m_s/n_s\} = w'}} (-1)^{|I|}$$

is identical with zero and so is

$$e^{2\pi i w} \sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x-v}{x-u}\Bigg) \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v\\ \{\Sigma_{s\in I} q_s m_s/n_s\} = w}} (-1)^{|I|},$$

thus

$$\begin{aligned} &\sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x-v}{x-u}\Bigg) \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v}} (-1)^{|I|} \\ &= \sum_{w\in W(\theta)} \sum_{v\in V(\theta)} \Bigg(\prod_{\substack{x\in U(\theta)\\ x\neq u}} \frac{x-v}{x-u}\Bigg) \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v\\ \{\Sigma_{s\in I} q_s m_s/n_s\} = w}} (-1)^{|I|} = 0. \end{aligned}$$

With the help of Lemma 7,

$$\sum_{v\in V(\theta)} v^j \sum_{\substack{I\subseteq J\\ \Sigma_{s\in I} m_s/n_s = v}} (-1)^{|I|} = 0 \quad \text{for every } j = 0, 1, \cdots, m-1.$$

For each $l = 0, 1, \cdots, m-1$ it follows that

$$\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\left[\sum_{s\in I}\frac{m_s}{n_s}\right]^l = \sum_{v\in V(\theta)}\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v}}(-1)^{|I|}(v-\theta)^l$$
$$=\sum_{j=0}^{l}\binom{l}{j}(-\theta)^{l-j}\sum_{v\in V(\theta)}v^j\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I}m_s/n_s=v}}(-1)^{|I|}=0.$$

This proves the second assertion in Ib.

Let's now proceed to part (II).

IIa) Suppose that $m > |\{1 \leqslant s \leqslant k:\ n_s \mid m_s\}| \geqslant |\{s \in J:\ n_s \mid m_s\}|$, then $\{(n_s/m_s)\mathbb{Z}\}_{s\in J}$ doesn't cover 1 at least m times. By Lemma 4, for some $\theta \in S$,

$$\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}=\theta}} (-1)^{|I|}\binom{[\Sigma_{s\in I}m_s/n_s]}{l}e^{\Sigma_{s\in I}0\cdot m_s/n_s}\neq 0\ \text{ for some } l=0,1,\cdots,m-1$$

and therefore (45) follows from Lemma 6. By part Ib, $|P(\theta)| \geqslant |W(\theta)| \geqslant d(\sigma-\theta)$ for some $\sigma \in S\setminus\{\theta\}$. For $v = \theta + \Sigma_{s\in\{1,\cdots,k\}\setminus J}m_s/n_s$,

$$|K(v)| \geqslant \left|\left\{I\cup(\{1,\cdots,k\}\setminus J):\ I\subseteq J\ \&\ \sum_{s\in I}\frac{m_s}{n_s}+\sum_{s\in\{1,\cdots,k\}\setminus J}\frac{m_s}{n_s}-v\in\mathbb{Z}\right\}\right|$$
$$=\left|\left\{I\subseteq J:\ \sum_{s\in I}\frac{m_s}{n_s}-\theta\in\mathbb{Z}\right\}\right| = |P(\theta)|$$

and so for some $I_0 \subseteq J$ with $\{\Sigma_{s\in I_0}m_s/n_s\} \neq \theta$ we have

$$|K(v)| \geqslant d\left(\left\{\sum_{s\in I_0}\frac{m_s}{n_s}\right\}-\theta\right) = d\left(\sum_{s\in I_0}\frac{m_s}{n_s}-v+\sum_{\substack{s=1\\ s\notin J}}^{k}\frac{m_s}{n_s}\right) = d\left(\sum_{s\in I_1}\frac{m_s}{n_s}-v\right)$$

where $I_1 = I_0 \cup (\{1,\cdots,k\}\setminus J) \subseteq \{1,\cdots,k\}$ and $\Sigma_{s\in I_1}m_s/n_s - v \notin \mathbb{Z}$.

IIb) Let $J' \subseteq \{1,\cdots,k\}\setminus J$ and $v' = v - \Sigma_{s\in J'}m_s/n_s$. If $I \subseteq J$ then

$$\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}-\{v'\} \equiv \sum_{s\in I}\frac{m_s}{n_s}-v+\sum_{s\in J'}\frac{m_s}{n_s} \equiv \sum_{s\in I\cup J'}\frac{m_s}{n_s}-v \pmod 1.$$

We claim that

$$|V(\{v'\})| = \left|\left\{\sum_{s\in I}\frac{m_s}{n_s}:\ I\subseteq J\ \&\ \sum_{s\in I}\frac{m_s}{n_s}+\sum_{s\in J'}\frac{m_s}{n_s}-v\in\mathbb{Z}\right\}\right|$$
$$=\left|\left\{\sum_{s\in I\cup J'}\frac{m_s}{n_s}:\ I\subseteq J\ \&\ \sum_{s\in I\cup J'}\frac{m_s}{n_s}-v\in\mathbb{Z}\right\}\right| \leqslant m.$$

For, if not, then $|\{\Sigma_{s\in I}m_s/n_s:\ I \in K(v)\}| \not\leqslant m$, $\Sigma_{s\in I}m_s/n_s \geqslant m + \{v'\}$ for some $I \subseteq J$, $m_s = (n, n_s)$ for each $s = 1,\cdots,k$, $\Sigma_{s\in I'}(n,n_s)/n_s - v \notin \mathbb{Z}$ for all $I' \subseteq \{1,\cdots,k\}\setminus\{t\}$ and A covers $a+n\mathbb{Z}$ exactly m times with $a_t + n_t\mathbb{Z}$

essential, hence $t \in J$, $v' \notin \mathbb{Z}$, system $A' = \{q_s + (n_s/(n, n_s))\mathbb{Z}\}_{s\in J}$ forms an exact m-cover of $\mathbb{Z}$ (by Lemma 1) and therefore

$$m = \sum_{s\in J} \frac{1}{n_s/(n, n_s)} \geqslant \sum_{s\in I} \frac{(n, n_s)}{n_s} \geqslant m + \{v'\} > m,$$

a contradiction!

As

$$\begin{aligned}|L(v)| &= \left| \bigcup_{J'\subseteq\{1,\cdots,k\}\setminus J} \left\{ I \cup J' :\ I \subseteq J \ \&\ \sum_{s\in I\cup J'} \frac{m_s}{n_s} = v \right\} \right| \\ &= \sum_{J'\subseteq\{1,\cdots,k\}\setminus J} \left| \left\{ I \subseteq J :\ \sum_{s\in I} \frac{m_s}{n_s} = v - \sum_{s\in J'} \frac{m_s}{n_s} \right\} \right|,\end{aligned}$$

provided that $2 \nmid |L(v)|$, for some $J' \subseteq \{1, \cdots, k\} \setminus J$, $|\{I \subseteq J :\ \Sigma_{s\in I} m_s/n_s = v'\}|$ is odd where $v' = v - \Sigma_{s\in J'} m_s/n_s$, and hence for some $w \in W(\{v'\})$ we have

$$\begin{aligned}&\left| \left\{ I \subseteq J :\ 2 \mid |I|,\ \sum_{s\in I} \frac{m_s}{n_s} = v' \ \&\ \left\{ \sum_{s\in I} \frac{m_s}{n_s} q_s \right\} = w \right\} \right| \\ \neq &\left| \left\{ I \subseteq J :\ 2 \nmid |I|,\ \sum_{s\in I} \frac{m_s}{n_s} = v' \ \&\ \left\{ \sum_{s\in I} \frac{m_s}{n_s} q_s \right\} = w \right\} \right|.\end{aligned}$$

Since $|V(\{v'\})| \leqslant m$, by part Ia there exists an $I' \subseteq J$ with $\{\Sigma_{s\in I'} m_s/n_s\} \neq \{v'\}$ such that

$$\begin{aligned}|L(v)| \geqslant &\left| \left\{ I \subseteq J :\ \sum_{s\in I} \frac{m_s}{n_s} = v' \right\} \right| \geqslant |W_{v'}(\{v'\})| \\ \geqslant &d\left(\left\{ \sum_{s\in I'} \frac{m_s}{n_s} \right\} - \{v'\} \right) = d\left(\sum_{s\in I''} \frac{m_s}{n_s} - v \right)\end{aligned}$$

where $I'' = I' \cup J' \subseteq \{1, \cdots, k\}$ and $\Sigma_{s\in I''} m_s/n_s - v \notin \mathbb{Z}$.

On the condition that there is an $\varepsilon \in \{1, -1\}$ for which $(-1)^{|I|} = \varepsilon$ for every $I \in L(v)$, for each $J' \subseteq \{1, \cdots, k\} \setminus J$ all the $(-1)^{|I|}$ with $I \subseteq J$ and $\Sigma_{s\in I} m_s/n_s = v - \Sigma_{s\in J'} m_s/n_s$ have the same value. Clearly it suffices to show that

$$\begin{aligned}|L(v)| = &\sum_{J'\subseteq\{1,\cdots,k\}\setminus J} \left| \left\{ I \subseteq J :\ \sum_{s\in I} \frac{m_s}{n_s} = v - \sum_{s\in J'} \frac{m_s}{n_s} \right\} \right| \\ \in &\left\{ \sum_{\substack{I\subseteq\{1,\cdots,k\} \\ \Sigma_{s\in I} m_s/n_s - v \notin \mathbb{Z}}} x_I d\left(\sum_{s\in I} \frac{m_s}{n_s} - v \right) :\ x_I \in \mathbb{N} \text{ for each } I \right\}.\end{aligned}$$

Fix $J' \subseteq \{1, \cdots, k\} \setminus J$. As $|V(\{v'\})| \leqslant m$ where $v' = v - \Sigma_{s\in J'} m_s/n_s$, by part Ib

$$\left| \left\{ I \subseteq J :\ \sum_{s\in I} \frac{m_s}{n_s} = v' \right\} \right| = \left| \sum_{\substack{I\subseteq J \\ \Sigma_{s\in I} m_s/n_s = v'}} (-1)^{|I|} \right|$$

can be written in the form

$$\sum_{\substack{I\subseteq J\\ \{\Sigma_{s\in I}m_s/n_s\}\neq\{v'\}}} c_I d\bigg(\Big\{\sum_{s\in I}\frac{m_s}{n_s}\Big\}-\{v'\}\bigg)=\sum_{\substack{I\subseteq J\\ \Sigma_{s\in I\cup J'}m_s/n_s-v\notin\mathbb{Z}}} c_I d\bigg(\sum_{s\in I\cup J'}\frac{m_s}{n_s}-v\bigg)$$

where all the c_I are nonnegative integers. We are done.

In contrast with Corollaries 1 and 2 we give here

Corollary 11. . *Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be an exact m-cover of some $AS(n)$ and J a subset of $\{1,\cdots,k\}$ with $v=\Sigma_{s\in J}(n,n_s)/n_s\notin\mathbb{Z}$. If A is minimal, or there is a $t\in\{1,\cdots,k\}$ with $a_t+n_t\mathbb{Z}$ essential such that $\Sigma_{s\in I}(n,n_s)/n_s-v\notin\mathbb{Z}$ for all $I\subseteq\{1,\cdots,k\}\setminus\{t\}$, then either*

$$(50)\qquad \begin{aligned}&\bigg|\bigg\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{(n,n_s)}{n_s}=v\bigg\}\bigg|\\ &\geqslant \min_{\substack{I\subseteq\{1,\cdots,k\}\\ \Sigma_{s\in I}(n,n_s)/n_s-v\notin\mathbb{Z}}} d\left(\sum_{s\in I}\frac{(n,n_s)}{n_s}-v\right)\geqslant p\left(\frac{[n,n_1,\cdots,n_k]}{n}\right),\end{aligned}$$

or

$$(51)\qquad \bigg|\bigg\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{(n,n_s)}{n_s}=v\bigg\}\bigg|\equiv 0 \pmod 2$$

and

$$(52)\qquad \sum_{s\in I}\frac{(n,n_s)}{n_s}=v \text{ for some } I\subseteq\{1,\cdots,k\} \text{ with } |I|\not\equiv|J| \pmod 2.$$

Proof. It follows from part IIb of Theorem 3 and the following observations.

(a) When A is minimal, by Lemma 1 for some suitable integers $q_1,\cdots,q_k$ system $\{q_s+(n_s/(n,n_s))\mathbb{Z}\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$, and therefore

$$\bigg|\bigg\{\sum_{s\in I}\frac{(n,n_s)}{n_s}:\ I\subseteq\{1,\cdots,k\}\ \&\ \sum_{s\in I}\frac{(n,n_s)}{n_s}-v\in\mathbb{Z}\bigg\}\bigg|\leqslant m$$

since $\Sigma_{s=1}^k(n,n_s)/n_s=m$ and $v\notin\mathbb{Z}$.

(b) $\Sigma_{s\in I}(n,n_s)/n_s-v\notin\mathbb{Z}$ for some $I\subseteq\{1,\cdots,k\}$, e.g. $\Sigma_{s\in\emptyset}(n,n_s)/n_s-v\notin\mathbb{Z}$. And not all of $n_1,\cdots,n_k$ divide n because $\Sigma_{s\in J}(n,n_s)/n_s\notin\mathbb{Z}$.

(c) If $I\subseteq\{1,\cdots,k\}$ and $\Sigma_{s\in I}(n,n_s)/n_s-v\notin\mathbb{Z}$, then

$$\begin{aligned}&d\left(\sum_{s\in I}\frac{(n,n_s)}{n_s}-v\right)=d\left(\sum_{s\in I}\left(\frac{[n,n_s]}{n}\right)^{-1}-\sum_{s\in J}\left(\frac{[n,n_s]}{n}\right)^{-1}\right)\\ \geqslant& p\left(\left[\frac{[n,n_1]}{n},\cdots,\frac{[n,n_k]}{n}\right]\right)=p\left(\frac{[n,n_1,\cdots,n_k]}{n}\right).\end{aligned}$$

Remark. Let (1) be an exact m-cover of $\mathbb{Z}$. Then for each $t=1,\cdots,k$ system A forms an exact m-cover of $a_t+n\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ essential. If there exists a unique $J\subseteq\{1,\cdots,k\}$ with $\Sigma_{s\in J}(n,n_s)/n_s=v$, then $\bar J=\{1,\cdots,k\}\setminus J$ is the unique $I\subseteq\{1,\cdots,k\}$ such that $\Sigma_{s\in I}(n,n_s)/n_s=\bar v$ where $\bar v=\Sigma_{s=1}^k(n,n_s)/n_s-v$, by Corollary 11 for any $t=1,\cdots,k$ there exists an $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ with $\Sigma_{s\in I}(n,n_s)/n_s-\bar v\in\mathbb{Z}$ (which is obvious if $\bar v\in\mathbb{Z}$) and hence $\Sigma_{s\in\bar I}(n,n_s)/n_s-v\in\mathbb{Z}$

where $t \in \bar{I} = \{1, \cdots, k\} \setminus I$. This proves part (i) of Theorem II. (Please compare it with part (iii) of Corollary 1.) Since A forms a minimal exact m-cover of $\mathbb{Z}$, also by Corollary 11, if $v = \Sigma_{s\in J} 1/n_s \notin \mathbb{Z}$ where $J \subseteq \{1, \cdots, k\}$ then (50), or (51) and (52) hold for $n = 1$.

Corollary 12. *Suppose that system $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ covers an $AP_n(|\{\{\Sigma_{s\in I} r_s\} : I \subseteq \{1, \cdots, k\}\}|)$ at least m times where the r_s are rationals with $r_s n_s/(n, n_s) \in \mathbb{Z}^+$ and*

$$r_1 \geqslant \cdots \geqslant r_{k-l} > r_{k-l+1} = \cdots = r_k \quad (0 \leqslant l \leqslant k). \tag{53}$$

Let $\lambda \in \mathbb{Z}$, $0 \leqslant \lambda \leqslant l$, and $\lambda < r_{k-l}/r_k$ if $l < k$. Then either

$$\left|\left\{\sum_{s\in I} r_s : \ I \subseteq \{1, \cdots, k\} \ \& \ \sum_{s\in I} r_s - \lambda r_k \in \mathbb{Z}\right\}\right| > m \tag{54}$$

and hence

$$\sum_{s=1}^{k-\lambda} r_s = \sum_{s=1}^{k} r_s - \lambda r_k \geqslant m - [\lambda r_k], \tag{55}$$

or $\binom{l}{\lambda}$ can be written as the sum of some denominators greater than 1 of the rationals $\Sigma_{s\in I} r_s - \lambda r_k$, $I \subseteq \{1, \cdots, k\}$.

Proof. Set $m_1 = n_1 r_1, \ \cdots, \ m_k = n_k r_k$ and

$$V_\lambda = \left\{\sum_{s\in I} r_s : \ I \subseteq \{1, \cdots, k\} \ \& \ \sum_{s\in I} r_s \equiv \lambda r_k \pmod 1\right\}.$$

Clearly $m_s \in \mathbb{Z}^+$ and $(n, n_s) \mid m_s$ for $s = 1, \cdots, k$. If $|V_\lambda| > m$ then

$$\sum_{s=1}^{k} r_s \geqslant \max_{r\in V_\lambda} r \geqslant m + \{\lambda r_k\}$$

and (55) follows.

Now assume that $|V_\lambda| \leqslant m$. If $I \subseteq \{1, \cdots, k\}$ and $\lambda r_k = \Sigma_{s\in I} m_s/n_s = \Sigma_{s\in I} r_s$, then we must have $I \subseteq \{s \in \mathbb{Z} : \ k - l < s \leqslant k\}$ (since $r_1 \geqslant \cdots \geqslant r_{k-l} > \lambda r_k$ if $l < k$) and therefore $|I| = \lambda$. By part IIb of Theorem 3,

$$\binom{l}{\lambda} = \left|\left\{I \subseteq \{1, \cdots, k\} : \ \sum_{s\in I} \frac{m_s}{n_s} = \lambda r_k\right\}\right| = d_1 + \cdots + d_t$$

where $d_1, \cdots, d_t$ are suitable numbers in the set

$$\begin{aligned} &\left\{d\left(\sum_{s\in I} \frac{m_s}{n_s} - \lambda r_k\right) : \ I \subseteq \{1, \cdots, k\} \ \& \ \sum_{s\in I} \frac{m_s}{n_s} - \lambda r_k \notin \mathbb{Z}\right\} \\ =&\left\{d\left(\sum_{s\in I} r_s - \lambda r_k\right) : \ I \subseteq \{1, \cdots, k\} \ \& \ \sum_{s\in I} r_s - \lambda r_k \notin \mathbb{Z}\right\}. \end{aligned}$$

This completes the proof.

Let (1) be an m-cover of $\mathbb{Z}$. Putting $\lambda = 0$ and $n = 1$ in Corollary 12 we obtain part (i) of Theorem I. In the case $\lambda = n = 1$ and $r_s = 1/n_s$ for $s = 1, \cdots, k$, Corollary 12 yields the latter assertion in part (iv) of Theorem I. If (1) forms an exact m-cover of $\mathbb{Z}$, then part (iv) of Theorem II follows from Corollary 12 since $\Sigma_{s=1}^k 1/n_s = m$.

As for the former assertion in part (iv) of Theorem I, here we present

Corollary 13. *Let $A = \{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$ with (7) where $0 < l < k$ and $ln_{k-l} < n_k$. Then for every positive integer $r < n_k/(ln_{k-l})$ we have*

$$\begin{aligned}
&\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k-l\},\ \sum_{s\in I}\frac{1}{n_s}-\frac{jr}{n_k}\in\mathbb{Z}\text{ for some } j=0,\cdots,l\right\}\right| \\
(56)\quad &\geqslant\left|\left\{x=\sum_{s\in I}\frac{1}{n_s}+\frac{jr}{n_k}:\ I\subseteq\{1,\cdots,k-l\},\ j\in\{0,1,\cdots,l\},\ \{x\}=\frac{lr}{n_k}\right\}\right| \\
&>m,
\end{aligned}$$

thus $\Sigma_{s=1}^{k-l}1/n_s \geqslant m$, and either there are at least m positive integers in the form $\Sigma_{s\in I}1/n_s$ with $I\subseteq\{1,\cdots,k-l\}$ or

$$\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\cdots,k-l\}\right\}\supseteq\left\{\frac{rr'}{n_k}:\ r\in\mathbb{N}\ \&\ r<\frac{n_k}{ln_{k-l}}\right\} \tag{57}$$

where each r' is a suitable integer among $1,\cdots,l$.

Proof. Let r be any positive integer less than $n_k/(ln_{k-l})$. Applying Corollary 12 with $\lambda = l$, $n = 1$, $r_1 = 1/n_1, \cdots, r_{k-l} = 1/n_{k-l}$ and $r_{k-l+1} = \cdots = r_k = r/n_k$, we get the second inequality of (56) which implies that

$$\sum_{s=1}^{k-l}\frac{1}{n_s}+\frac{lr}{n_k}\geqslant m+\frac{lr}{n_k},\quad \text{i.e}\quad \sum_{s=1}^{k-l}\frac{1}{n_s}\geqslant m.$$

The first inequality of (56) is apparent. In fact, if $\Sigma_{s\in I_1}1/n_s + ir/n_k$ and $\Sigma_{s\in I_2}1/n_s + jr/n_k$ are distinct rationals with fractional part lr/n_k where $I_1, I_2 \subseteq \{1,\cdots,k-l\}$ and $i,j\in\{0,1,\cdots,l\}$, then $\Sigma_{s\in I_1}1/n_s \neq \Sigma_{s\in I_2}1/n_s$ since otherwise we would have

$$\frac{ir}{n_k}=\left\{\frac{ir}{n_k}\right\}\neq\left\{\frac{jr}{n_k}\right\}=\frac{jr}{n_k}$$

which is impossible.

Note that $\Sigma_{s\in\emptyset}1/n_s = 0\in\mathbb{N}$. When

$$\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ \emptyset\neq I\subseteq\{1,\cdots,k-l\}\text{ and }\sum_{s\in I}\frac{1}{n_s}\in\mathbb{Z}\right\}\right|<m,$$

by (56) $\{\Sigma_{s\in I}1/n_s\} = jr/n_k$ for some $I\subseteq\{1,\cdots,k-l\}$ and $j = 1,\cdots,l$. The proof is ended.

Example 3. Let $n > 2$ be odd and $k = 2n-1$. Let $a_s = 2^{s-1}$ and $n_s = 2^s$ for $s = 1,\cdots,n-1$, $a_s = 2^{n-1}(s-n+1)$ and $n_s = 2^{s-n}n$ for $s = n,\cdots,k$. Since

$$\{2^0+2\mathbb{Z},\ 2+2^2\mathbb{Z},\ \cdots,\ 2^{n-2}+2^{n-1}\mathbb{Z},\ 2^{n-1}\mathbb{Z}\}$$

disjointly covers $\mathbb{Z}$ and $\{2^{n-1}t+2^{t-1}n\mathbb{Z}\}_{t=1}^n$ covers $\bigcup_{t=1}^n 2^{n-1}t+2^{n-1}n\mathbb{Z} = 2^{n-1}\mathbb{Z}$ with all the $2^{n-1}t+2^{t-1}n\mathbb{Z}$ essential, $A = \{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms a minimal cover of

$\mathbb{Z}$ with $n_1, \ldots, n_k$ distinct. By a trivial computation

$$\sum_{s=1}^{k} \frac{1}{n_s} = \frac{(2^{n-1}-1)n + 2^n - 1}{2^{n-1}n} = 1 + \frac{2^n - n - 1}{2^{n-1}n} < 1 + \frac{2}{n} < 2,$$

and $\Sigma_{s=1}^{k} 1/n_s > 1$ as is claimed by a conjecture of Erdös. In view of Corollary 13 we should have $\Sigma_{s=1}^{k-1} 1/n_s \geqslant 1$, which can be easily verified.

Because every natural number can be uniquely expressed as a sum of distinct powers of two,

$$\left\{\sum_{s\in I} \frac{1}{n_s} : \ I \subseteq \{1, \cdots, n-1\}\right\} = \left\{0, \frac{1}{2^{n-1}}, \cdots, \frac{2^{n-1}-1}{2^{n-1}}\right\},$$

$$\left\{\sum_{s\in I} \frac{1}{n_s} : \ I \subseteq \{n, \cdots, k-1\}\right\} = \left\{0, \frac{1}{2^{n-2}n}, \cdots, \frac{2^{n-1}-1}{2^{n-2}n}\right\},$$

and

$$\left\{\sum_{s\in I} \frac{1}{n_s} : \ I \subseteq \{n, \cdots, k\}\right\} = \left\{0, \frac{1}{2^{n-1}n}, \cdots, \frac{2^n-1}{2^{n-1}n}\right\}.$$

Observe that

$$\begin{aligned}&\left\{\left\{\sum_{s\in I} \frac{1}{n_s}\right\} : \ I \subseteq \{1, \cdots, k-1\}\right\} \\ &= \left\{\left\{\sum_{s\in I_1} \frac{1}{n_s} + \sum_{s\in I_2} \frac{1}{n_s}\right\} : \ I_1 \subseteq \{1, \cdots, n-1\},\ I_2 \subseteq \{n, \cdots, k-1\}\right\} \\ &= \left\{\left\{\frac{i}{2^{n-1}} + \frac{j}{2^{n-2}n}\right\} : \ i, j = 0, 1, \cdots, 2^{n-1}-1\right\} \\ &= \left\{\left\{\frac{in+2j}{2^{n-1}n}\right\} : \ i, j = 0, 1, \cdots, 2^{n-1}-1\right\}.\end{aligned}$$

If we write a natural number r less than $2^{n-1}n$ in the form $cn + d$ where $c, d \in \mathbb{N}$, $c < 2^{n-1}$ and $d < n$, then

$$in + 2j \equiv cn + d = r \pmod{2^{n-1}n} \quad \text{for some } i, j = 0, 1, \cdots, 2^{n-1}-1,$$

in fact we may let $i = c,\ j = d/2$ if $2 \mid d$; $i = c-1,\ j = (d+n)/2$ if $c > 0$ and $2 \nmid d$; $i = 2^{n-1}-1$ and $j = (d+n)/2$ if $c = 0$ and $2 \nmid d$. Thus

$$\left\{\left\{\sum_{s\in I} \frac{1}{n_s}\right\} : \ I \subseteq \{1, \cdots, k-1\}\right\} = \left\{0, \frac{1}{2^{n-1}n}, \cdots, \frac{2^{n-1}n-1}{2^{n-1}n}\right\},$$

and hence

$$\begin{aligned}\left\{\left\{\sum_{s\in I} \frac{1}{n_s}\right\} : \ I \subseteq \{1, \cdots, k\}\right\} &= \left\{\left\{\sum_{s\in I} \frac{1}{n_s}\right\} : \ I \subseteq \{1, \cdots, k-1\}\right\} \\ &= \left\{\frac{r}{n_k} : \ r = 0, 1, \cdots, n_k - 1\right\}\end{aligned}$$

as is implied by Corollary 5.

Write $2^n - 1$ in the form $nq + a$ where $a, q \in \mathbb{Z}$ and $0 \leqslant a < n$. Then $2 \leqslant q \leqslant (2^n-1)/n \leqslant 2^{n-1}-1$, and $a \geq 1$ since $n \nmid 2^n - 1$ (cf. F10 of [8]). For

$r=0,1,\cdots,(2^{n-1}-1)n+2^n-1$ with $\{r/n\}>a/n$, $0\leqslant[r/n]<2^{n-1}-1+q$ and therefore

$$\begin{aligned}
&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}=\frac{r}{2^{n-1}n}\right\}\right| \\
=&\left|\left\{I_1\cup I_2:\ I_1\subseteq\{1,\cdots,n-1\},\ I_2\subseteq\{n,\cdots,k\},\ \sum_{s\in I_1}\frac{1}{n_s}+\sum_{s\in I_2}\frac{1}{n_s}=\frac{r}{2^{n-1}n}\right\}\right| \\
=&\left|\left\{\langle i,j\rangle\in\mathbb{N}\times\mathbb{N}:\ i\leqslant 2^{n-1}-1,\ j\leqslant 2^n-1\ \&\ \frac{i}{2^{n-1}}+\frac{j}{2^{n-1}n}=\frac{r}{2^{n-1}n}\right\}\right| \\
=&\left|\left\{\langle i,j\rangle\in\mathbb{Z}\times\mathbb{Z}:\ 0\leqslant i<2^{n-1},\ n\mid j-r,\ 0\leqslant\left[\frac{j}{n}\right]=\left[\frac{r}{n}\right]-i<q\right\}\right| \\
=&\left|\left\{i\in\mathbb{Z}:\ \max\left\{0,\ \left[\frac{r}{n}\right]-q+1\right\}\leqslant i\leqslant\min\left\{2^{n-1}-1,\ \left[\frac{r}{n}\right]\right\}\right\}\right| \\
=&\min\left\{2^{n-1}-1,\ \left[\frac{r}{n}\right]\right\}-\max\left\{0,\ \left[\frac{r}{n}\right]-q+1\right\}+1 \\
=&\min\left\{\left[\frac{r}{n}\right]+1,\ q,\ 2^{n-1}-1+q-\left[\frac{r}{n}\right]\right\} \\
=&\begin{cases}[r/n]+1, & \text{if } 0\leqslant[r/n]<q;\\ q, & \text{if } q\leqslant[r/n]<2^{n-1};\\ q-1-([r/n]-2^{n-1}), & \text{if } 2^{n-1}\leqslant[r/n]<2^{n-1}-1+q\end{cases} \\
>&0.
\end{aligned}$$

In a similar way, for any $r=0,1,\cdots,(2^{n-1}-1)n+2^n-1$ with $\{r/n\}\leqslant a/n$ one can deduce that

$$\begin{aligned}
&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}=\frac{r}{2^{n-1}n}\right\}\right| \\
=&\left|\left\{\langle i,j\rangle\in\mathbb{N}\times\mathbb{N}:\ i\leqslant 2^{n-1}-1,\ j\leqslant 2^n-1\ \&\ in+j=r\right\}\right| \\
=&\left|\left\{\langle i,j\rangle\in\mathbb{Z}\times\mathbb{Z}:\ 0\leqslant i<2^{n-1},\ n\mid j-r\ \&\ 0\leqslant\left[\frac{r}{n}\right]-i=\left[\frac{j}{n}\right]\leqslant q\right\}\right| \\
=&\left|\left\{i\in\mathbb{Z}:\ \max\left\{0,\ \left[\frac{r}{n}\right]-q\right\}\leqslant i\leqslant\min\left\{2^{n-1}-1,\ \left[\frac{r}{n}\right]\right\}\right\}\right| \\
=&\min\left\{2^{n-1}-1,\ \left[\frac{r}{n}\right]\right\}-\max\left\{0,\ \left[\frac{r}{n}\right]-q\right\}+1 \\
=&\min\left\{\left[\frac{r}{n}\right],\ q,\ 2^{n-1}-1+q-\left[\frac{r}{n}\right]\right\}+1 \\
=&\begin{cases}[r/n]+1, & \text{if } 0\leqslant[r/n]<q;\\ q+1, & \text{if } q\leqslant[r/n]<2^{n-1};\\ q-([r/n]-2^{n-1}), & \text{if } 2^{n-1}\leqslant[r/n]\leqslant 2^{n-1}-1+q\end{cases} \\
>&0,
\end{aligned}$$

in particular

$$\begin{aligned}&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}=1\right\}\right|\\ =&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}=1+\frac{1}{2^{n-1}n}\right\}\right|\\ =&q=\left[\frac{2^n-1}{n}\right].\end{aligned}$$

Thus

$$\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k\}\right\}=\left\{\frac{r}{2^{n-1}n}:\ r=0,1,\cdots,(2^{n-1}-1)n+2^n-1\right\},$$

$$\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}=\frac{r}{2^{n-1}n}\right\}\right|=1$$

for $r\in\{0,1,\cdots,n-1\}\cup\{(2^{n-1}-2)n+2^n,\cdots,(2^{n-1}-1)n+2^n-1\}$,

$$\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{1}{n_s}\in\mathbb{Z}^+\right\}\right|>0$$

as is told by Zhang's result or part (i) of Theorem I, and

$$\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k\}\ \&\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{1}{n_k}\right\}\right|=\left|\left\{\frac{1}{n_k},1+\frac{1}{n_k}\right\}\right|>1$$

as is asserted by Corollary 13 since $n_{k-1}<n_k=2n_{k-1}$.

Remark. J. D. Swift ([23]) noted in 1954 that if $g\equiv 2 \pmod 4$ is a primitive root modulo an odd prime p then

$$\{g^0+2\mathbb{Z},g+2^2\mathbb{Z},\cdots,g^{p-2}+2^{p-1}\mathbb{Z},g^0+2^0p\mathbb{Z},g+2p\mathbb{Z},\cdots,g^{p-2}+2^{p-2}p\mathbb{Z},2^{p-1}p\mathbb{Z}\}$$

forms a cover of $\mathbb{Z}$. This can be easily obtained from Example 3 with $n=p$.

Putting $n=3$ in Example 3 we get the following cover of $\mathbb{Z}$:

$$\{1+2\mathbb{Z},\ 1+3\mathbb{Z},\ 2+4\mathbb{Z},\ 2+6\mathbb{Z},\ 12\mathbb{Z}\}.$$

It can be shown that if $\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $k>1$ and $n_1<\cdots<n_k$ then $k\geqslant 5$ and $n_k\geqslant 12$ (cf. [10]).

Let $x\geq 12$ and set

$$D(x)=\inf\sum_{s=1}^k\frac{1}{n_s}\tag{58}$$

where the infimum is taken over all those covers (1) of $\mathbb{Z}$ with $k>1$ and $n_1<\cdots<n_k\leqslant x$. An unsolved problem is to determine $D(x)$ (see [13]). Let n be the odd integer with

$$1<\frac{\log x}{2.5\log 2}\leqslant n<2+\frac{\log x}{2.5\log 2},$$

by Example 3

$$D(x) < 1 + \frac{2}{n} \leqslant 1 + \frac{5\log 2}{\log x}$$

since $2^{3-1} \cdot 3 = 12 \leqslant x$, and $2^{n-1}n \leqslant 2^{n-1} \cdot 2^{1.5n-4} = 2^{2.5(n-2)} < x$ if $n \geqslant 4$. Note that when $\{a_s + n_s\mathbb{Z}\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ with $k > 1$ and $n_1 < \cdots < n_k$, by Corollary 13

$$\sum_{s=1}^{k} \frac{1}{n_s} = \sum_{s=1}^{k-1} \frac{1}{n_s} + \frac{1}{n_k} \geqslant 1 + \frac{1}{n_k}.$$

So we have

$$\frac{1}{x} \leqslant D(x) - 1 < \frac{5\log 2}{\log x} \quad \text{for } x \geqslant 12 \tag{59}$$

and therefore

$$D(x) = 1 + O\left(\frac{1}{\log x}\right), \quad \lim_{x\to+\infty} D(x) = 1. \tag{60}$$

6. Open Problems and New Conjectures

The contents of the previous sections suggest some further questions and new conjectures.

Problem 1. Find combinatorial proofs of the theorems and corollaries which have no direct connections with the roots of unity. Extend some of our results to infinite (exact) m-covers of $\mathbb{Z}$. Provide something analogous to Theorems I and II for (exact) m-covers of the squares with the help of quadratic Gauss sums.

The observation that an arithmetic sequence $a + n\mathbb{Z}$ can be viewed as a coset of an ideal in the ring of rational integers yields

Problem 2. Let O_K be the ring of algebraic integers in an algebraic number field K. Let $a_1, \cdots, a_k \in O_K$ and $A_1, \cdots, A_k$ be integral ideals of O_K whose norms (with respect to the field extension $K/\mathbb{Q}$) are $n_1, \cdots, n_k$ respectively. On the condition that $\{a_s + A_s\}_{s=1}^k$ forms an (exact) m-cover of O_K, will parts (i)$-$(iv) of Theorem I (resp., Theorem II) still hold? We conjecture the positive answer for those K with class number 1.

Note that $\mathbb{Z}$ is an infinite cyclic group (under the usual addition) for which $n\mathbb{Z}$ is its subgroup of index n. In general one may study (finite) m-covers of a group by left cosets of its subgroups. As a matter of fact, some known results introduced in Section 1 have already been generalized in this direction. However there is no obvious way to attack

Problem 3. Let G be a group and $G_1, \cdots, G_k$ its subgroups of indices $n_1, \cdots, n_k$ respectively. Provided that $\{a_sG_s\}_{s=1}^k$ is an (exact) m-cover of G for some elements $a_1, \cdots, a_k$ of G, whether parts (i)$-$(iv) of Theorem I (resp., Theorem II) remain true? We conjecture that this is the case if $G_1, \cdots, G_k$ are subnormal in G.

Now we present a problem as a supplement to Theorem 1.

Problem 4. If we replace 'at least' and 'm-cover' in the first part of Theorem 1 by 'exactly' and 'exact m-cover' respectively, will the new version of part (i) of Theorem 1 be valid? In particular, when $\{a_s + n_s\mathbb{Z}\}_{s=1}^k$ covers $|\{\{\Sigma_{s\in I} 1/n_s\} : I \subseteq$

$\{1,\cdots,k\}\}|$ consecutive integers exactly m times, does it necessarily form an exact m-cover of $\mathbb{Z}$? We believe so.

Crittenden and Vanden Eynden conjectured in 1972 that if $n_1,\cdots,n_k$ are not less than a positive integer l not exceeding k, then $\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ whenever it covers $1,\cdots,2^{k-l+1}l$. (See E23 of [8].) This together with part (i) of Theorem 1 suggests

Problem 5. Are there any integers $m_1,\cdots,m_k$ prime to $n_1,\cdots,n_k$ respectively such that

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\leqslant 2^{k-l+1}l$$

where $l=\min\{k,n_1,\cdots,n_k\}$? If so, the Crittenden–Vanden Eynden conjecture will follow from Theorem 1.

In contrast with Lemma 9, which is a useful tool, we propose

Problem 6. If $c_1,\cdots,c_k$ and d are integers for which there exist nonzero numbers $x_1,\cdots,x_k$ such that $\Sigma_{s=1}^k c_s x_s^t=0$ for all those $t\in\mathbb{Z}^+$ not divisible by d, does d divide $c_1+\cdots+c_k$? We conjecture that the answer is affirmative.

The following problem is challenging and fascinating.

Problem 7. Characterize those tuples $\{n_s\}_{s=1}^k$ such that for each $t=1,\cdots,k$ and $r=0,1,\cdots,n_t-1$ there exists an $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ with $\Sigma_{s\in I}1/n_s\equiv r/n_t \pmod 1$. For what kind of covers of $\mathbb{Z}$ do the common differences form such a tuple? We conjecture that if $\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ forms an m-cover of $\mathbb{Z}$ and an exact m-cover of $a_t+n_t\mathbb{Z}$ with $1\leqslant t\leqslant k$ then for any $r=0,1,\cdots,n_t-1$ there is an $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ such that $\Sigma_{s\in I}1/n_s\equiv r/n_t \pmod 1$.

Let's conclude the paper with

Problem 8. Is it true that $D(x)\geqslant 1+c_1/\log x$ for $x\geqslant 12$? Can we have $D(x)-1\sim c_2/\log x$ as $x\to+\infty$? Here $D(x)$ is as in (58) and c_1,c_2 are suitable positive constants.

Acknowledgements. The author is indebted to the referee for his helpful suggestions. The revision was done during the author's visit in Italy, he would like to thank Trento University for its financial support.

References

1. M. A. Berger, A. Felzenbaum and A. S. Fraenkel, *Disjoint covering systems with precisely one multiple modulus*, Acta Arith. **50** (1988), 171–182. MR **89i:**11009
2. N. Burshtein, *On natural exactly covering systems of congruences having moduli occurring at most M times*, Discrete Math. **14** (1976), 205–214. MR **53:**2886
3. Y. G. Chen and Š. Porubský, *Remarks on systems of congruence classes*, Acta Arith. **71** (1995), 1–10. CMP 95:14
4. C. Cobeli, M. Vâjâitu and A. Zaharescu, *Graham's conjecture under Riemann hypothesis*, J. Number Theory **31** (1989), 80–87. MR **90c:**11064
5. R. B. Crittenden and C. L. Vanden Eynden, *Any n arithmetic progressions covering the first 2^n integers cover all integers*, Proc. Amer. Math. Soc. **24** (1970), 475–481. MR **41:**3365
6. P. Erdös, *On integers of the form 2^k+p and some related problems*, Summa Brasil. Math. **2** (1950), 113–123. MR **13:**437
7. P. Erdös, *Remarks on number theory IV: Extremal problems in number theory I*, Mat. Lapok **13** (1962), 228-255. MR **33:**4020

8. R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York, 1981. MR **83k:**10002
9. N. Jacobson, *Basic Algebra I (2nd Ed.)*, W. H. Freeman and Company, New York, 1985, p. 140. MR **86d:**00001
10. C. E. Krukenberg, *Covering sets of the integers*, Ph. D. thesis, University of Illinois (1971).
11. M. Newman, *Roots of unity and covering sets*, Math. Ann. **191** (1971), 279–282. MR **44:**3972
12. Š. Porubský, *On m times covering systems of congruences*, Acta Arith. **29** (1976), 159–169. MR **53:**2884
13. Š. Porubský, *Results and Problems on Covering Systems of Residue Classes*, Mitt. Math. Sem. Giessen, Heft 150, Giessen Univ., 1981, pp. 1–85. MR **83b:**10068
14. A. Schinzel, *Reducibility of polynomials and covering systems of congruences*, Acta Arith. **13** (1967), 91–101. MR **36:**2596
15. R. J. Simpson, *Regular coverings of the integers by arithmetic progressions*, Acta Arith. **45** (1985), 145–152. MR **86j:**11004
16. R. J. Simpson, *Exact coverings of the integers by arithmetic progressions*, Discrete Math. **59** (1986), 181–190. MR **87f:**11011
17. Z. W. Sun, *Finite coverings of groups*, Fund. Math. **134** (1990), 37–53. MR **91g:**20031
18. Z. W. Sun, *A theorem concerning systems of residue classes*, Acta Math. Univ. Comenian. **60** (1991), 123-131. MR **92f:**11007
19. Z. W. Sun, *An improvement to the Znám-Newman result*, Chinese Quart. J. Math. **6** (3) (1991), 90–96.
20. Z. W. Sun, *On exactly m times covers*, Israel J. Math. **77** (1992), 345–348. MR **93k:**11007
21. Z. W. Sun, *Covering the integers by arithmetic sequences*, Acta Arith. **72** (1995), 109–129. CMP 95:17
22. Z. W. Sun and Z. H. Sun, *Some results on covering systems of congruences*, J. Southwest-China Teachers Univ. (1) (1987), 10–15. Zbl. M. 749.11018.
23. J. D. Swift, *Sets of covering congruences*, Bull. Amer. Math. Soc. **60** (1954), 390.
24. M. Szegedy, *The solution of Graham's greatest common divisor problem*, Combinatorica **6** (1986), 67–71. MR **87i:**11010
25. M. Z. Zhang, *A note on covering systems of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **26** (1989), Special Issue, 185–188. MR **92c:**11003
26. M. Z. Zhang, *On irreducible exactly m times covering system of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **28** (1991), 403–408. MR **92j:**11001
27. Š Znám, *On Mycielski's problem on systems of arithmetic progressions*, Colloq. Math. **15** (1966), 201–204. MR **34:**134
28. Š. Znám, On exactly covering systems of arithmetic sequences, *in: Number Theory (Colloq. János Bolyai Math. Soc., Debrecen 1968)*, North-Holland, Amsterdam, 1970, pp. 221–225. MR **42:**7586
29. Š. Znám, *On properties of systems of arithmetic sequences*, Acta Arith. **26** (1975), 279–283. MR **51:**329

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Dipartimento di Matematica, Università degli Studi di Trento, I-38050 Povo (Trento), Italy

E-mail address: `zhiwei@science.unitn.it`